Field Factor of two Similar Cylindrical Conductors

By

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Abstract

A method is described in which the expression for the voltage at any point in a system of adjacent cylindrical conductors is obtained with the simple assumption that the length of the cylinders is very much bigger than the radius. Without further assumption and using this derived expression, simple expressions are then given for the average field and maximum field from which the field factor is easily calculated. The calculated values of the field factor agree with published tables.

Introduction

The paper starts with the fundamental consideration of the field around the simple isolated line charge. This gives the result that the equipotential surfaces for the isolated line charge are cylindrical concentric with the line.

Using the derived equations for the single line charge a second line charge is added to the system to form a single-phase system. The electric field at any point is derived and the equipotential surfaces are found to be cylindrical but these are not concentric with the line charges.

In order to utilise the above consideration for the single-phase system of cylindrical conductors of the same radius, line charges are substituted for the conductors. The cylinder surfaces of the conductors are equipotentials to these line charges. All the equations for the line charges were used to analyse the two cylinder system.

Having fully analysed the cylindrical system, simple expressions for the maximum field stress are derived as well as the mean value of the stress. The field factor, which is the ratio of mean stress to the maximum stress, is calculated.

List of Symbols

	—	radius of cylindrical conductor
s	—	shortest distance between the two cylindrical conductor surfaces
	—	permittivity of dielectric
a	—	distance between the line charge and the centre of a cylindrical conductor in a single phase system
σ	—	distance between the line charge and the surface of a cylindrical conductor surface in a single phase system measured along the line joining the centres of the conductors
x	—	distance between the cylindrical conductor surface and an equipotential surface measured along the line joining the two centres of the conductors
K	—	distance between the two equivalent line charges
p	—	$\frac{s + r}{r}$, a dimensionless constant
	—	$\frac{E_{mean}}{E_{max}}$, the field factor
m	—	$\frac{k}{\sigma}$

1. Field due to a single line charge:

Consider a line charge of length as shown in Fig. 1.

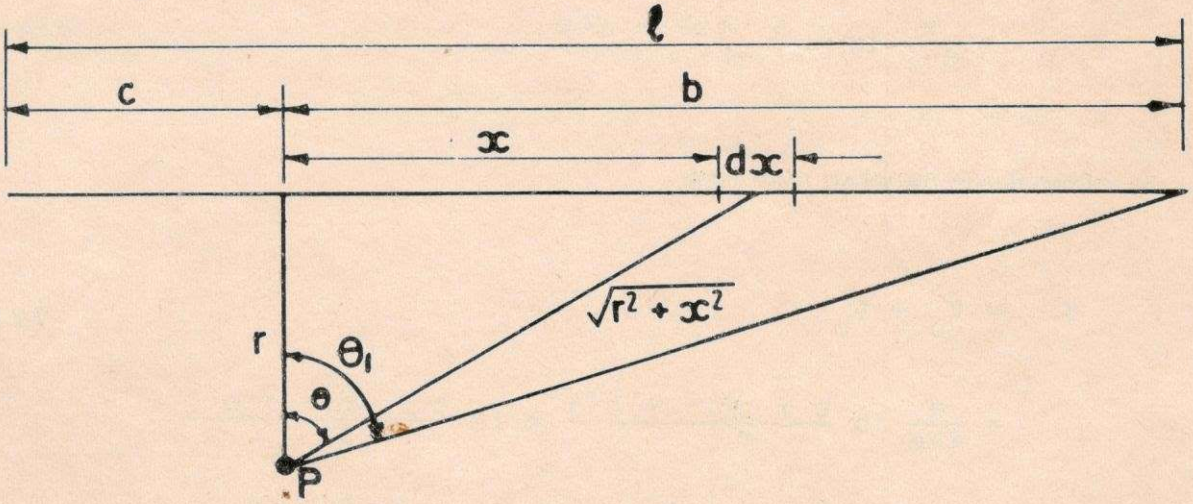


Fig. 1.

Then the potential, dV, due to an element of charge q dx at point ρ is:

$$dV = \frac{q}{4\pi\varepsilon\sqrt{(r^2 + x^2)}}\, dx \qquad (1)$$

where q is the charge per metre length of the uniformly charged line. Then potential at P due to length b is

$$V_b = \int_0^b \frac{q\,dx}{4\pi\varepsilon\ \sqrt{(r^2 + x^2)}} \qquad (2)$$

also $x = r \tan\theta$

$\therefore\ dx = r \sec^2\theta\ d\theta$ (3)

$$\therefore\ V_b = \frac{q}{4\pi\varepsilon} \int_0^{\tan^{-1}\frac{b}{r}} \cdot \frac{r \sec^2\theta\ d\theta}{\sqrt{(r^2 + r^2 \tan^2\theta)}} \qquad (4)$$

$$\doteq \frac{q}{4\pi\varepsilon} \int_0^{\tan^{-1}\frac{b}{r}} \sec\theta\ d\theta$$

$$= \frac{q}{4\pi\varepsilon} \left[\ln(\sec\theta + \tan\theta)\right]_0^{\tan^{-1}\frac{b}{r}}$$

By putting the limits $\theta = \tan^{-1}\frac{b}{r}$ then

$\tan\theta = \frac{b}{r}$

and $\sec\theta = \frac{\sqrt{(r^2 + b^2)}}{r}$

$$\therefore\ V_b = \frac{q}{4\pi\varepsilon} \ln \frac{b + \sqrt{(r^2 + b^2)}}{r} \qquad (5)$$

Similarly the potential due to length c of the line charge

$$V_c = \frac{q}{4\pi\varepsilon} \ln \frac{c + \sqrt{(r^2 + c^2)}}{r} \qquad (6)$$

thus the voltage due to the whole line charge

$$V = V_c + V_b \qquad (7)$$

$$= \frac{q}{4\pi\varepsilon} \ln \frac{b + \sqrt{(r^2 + b^2)}}{r} + \ln \frac{c + \sqrt{(r^2 + c^2)}}{r}$$

$$= \frac{q}{4\pi\varepsilon} \ln \frac{\{b + \sqrt{(r^2 + b^2)}\}\{c + \sqrt{(r^2 + c^2)}\}}{r^2}$$

Now for very long lines $r << b$ and $r << c$ thus

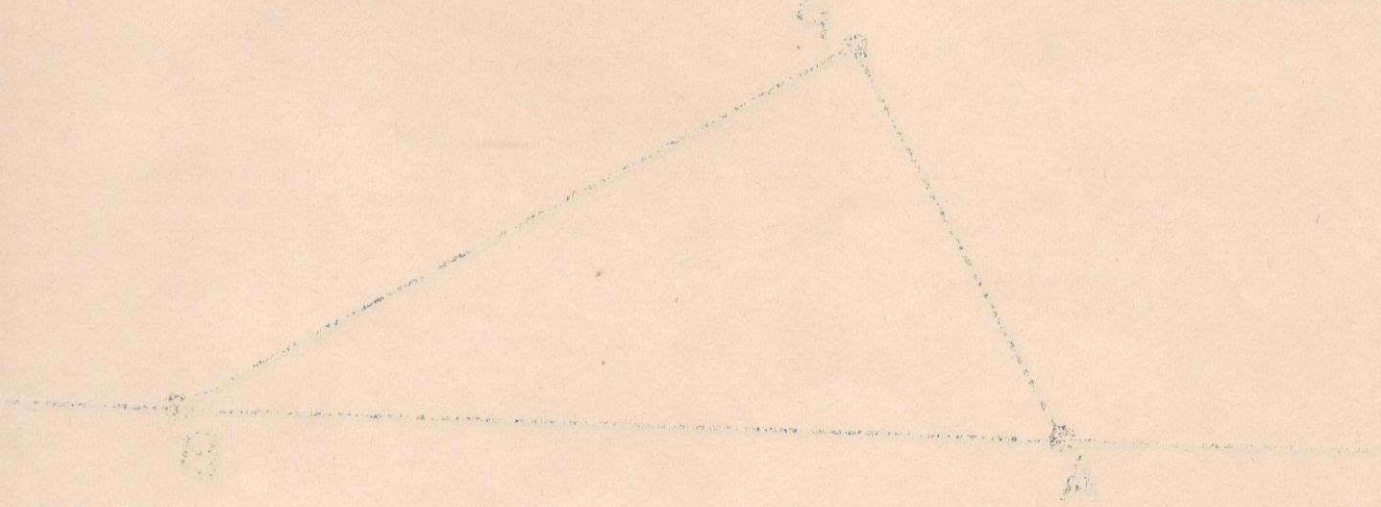
$$V = \frac{q}{4\pi\varepsilon} \ln \frac{\{ b + b(1+\frac{r^2}{2b^2} + \ldots)\} \{ c + c(1+\frac{r^2}{2c^2} + \ldots)\}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon} \ln \{ 2b + \frac{br^2}{2b^2} + \ldots\} \{ 2c + \ldots) \}$$

$$= \frac{q}{4\pi\varepsilon} \ln 4cb \frac{\{ 1 + \frac{r^2}{4b^2} + \ldots\}\{ 1 + \frac{r^2}{4c^2} + \ldots \}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon} \ln \frac{4cb}{r^2} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (8)$$

neglecting terms in $\frac{r^2}{b^2}$ and $\frac{r^2}{c^2}$



2. Two Parallel Line Charges

Consider two parallel line charges the plan of which is shown in Fig. 2.

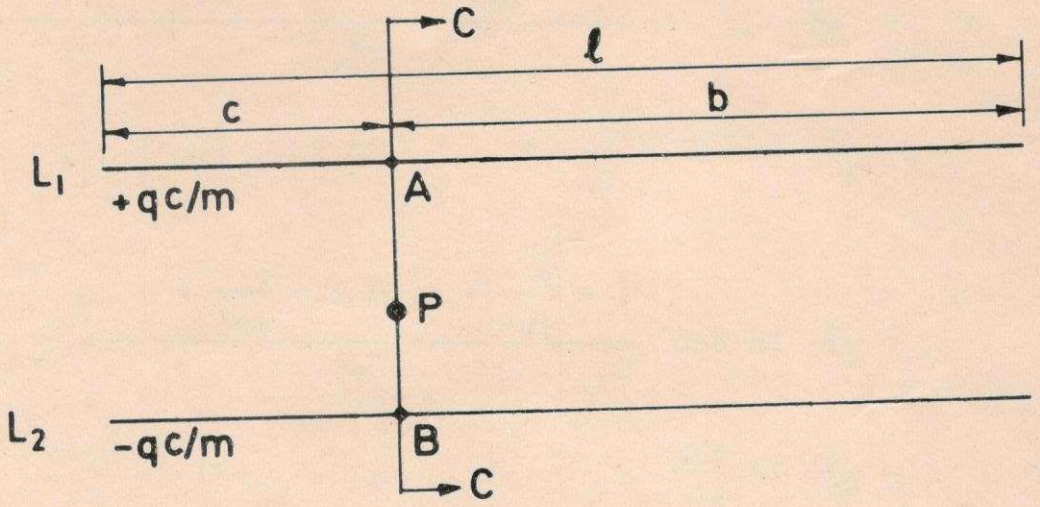


Fig. 2

and the section CC is shown in Fig. 3.

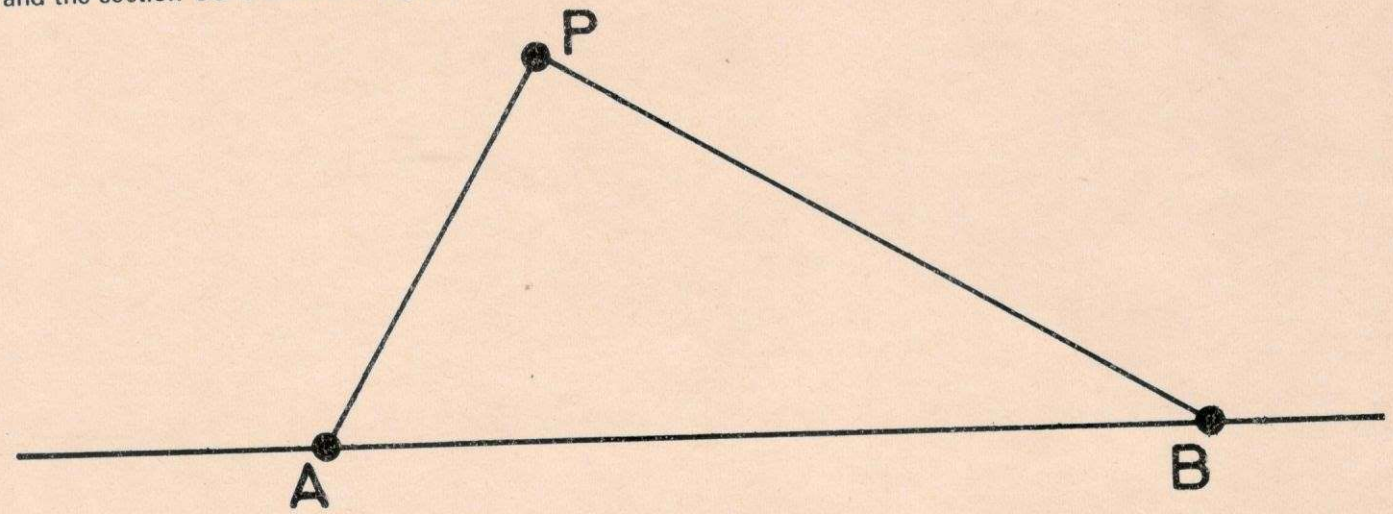


Fig. 3

From equation (8) then

$$V_p = \frac{q}{4\pi\varepsilon}\left[\ln\frac{4cb}{AP^2} - \ln\frac{4cb}{BP^2}\right]$$

$$= \frac{q}{4\pi\varepsilon}\ln\frac{BP^2}{AP^2}$$

$$= \frac{q}{2\pi\varepsilon}\ln\frac{BP}{AP} \quad \ldots \quad (9)$$

Consider the construction in Fig. 4.

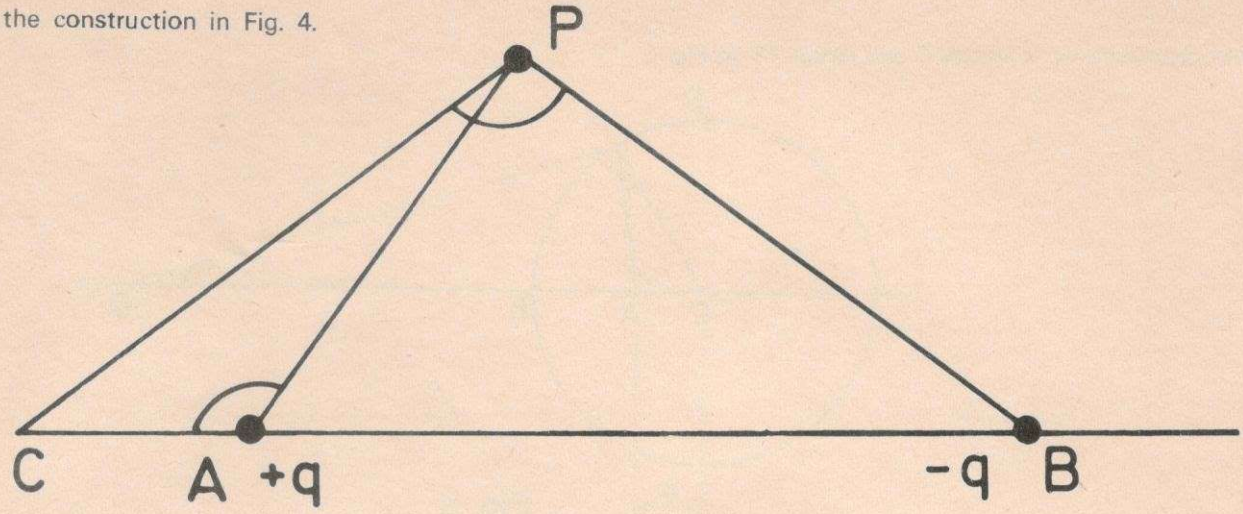


Fig. 4

such that $\angle$ BPC = $\angle$ PAC (10)

then triangles PAC and PCB are similar since

$$\angle BPC = \angle PAC \text{ by construction}$$

$$\text{Now } \angle PAC = \angle PBA + \angle APB$$

$$\text{and } \angle PAC = \angle BPC = \angle APC + \angle APB$$

$$\therefore \angle PBA = \angle APC \quad (11)$$

since the three angles of the two triangles PAC and PCB are equal then they are similar as shown in Fig. 5 below

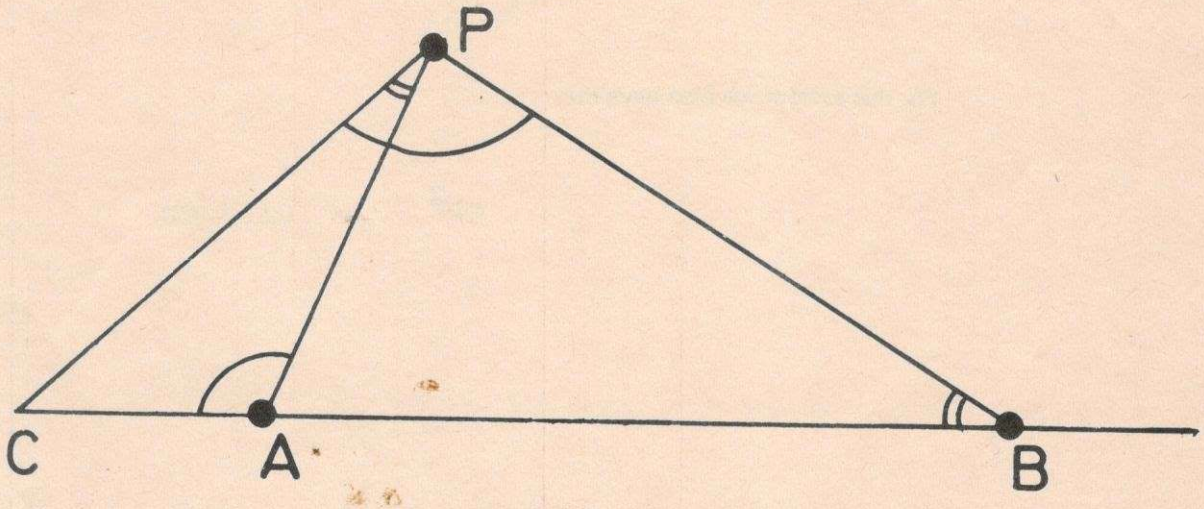


Fig. 5

$$\text{then } \frac{BP}{AP} = \frac{CP}{CA} = \frac{CB}{CP} \quad (12)$$

Next consider a circle centre C and radius CP in Fig. 6.

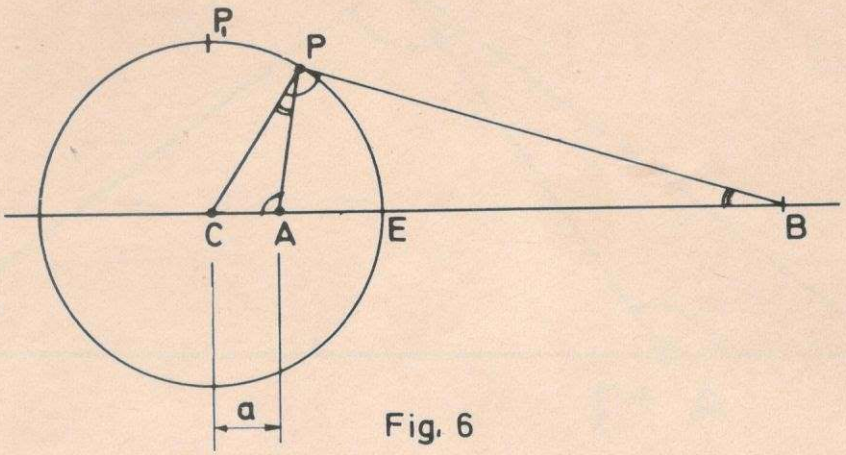


Fig. 6

then for any point P on this circle, using equation (12) we have

$$\frac{BP_1}{AP_1} = \frac{CP_1}{CA} = \frac{CB}{CP_1} \qquad (13)$$

but $CP = CP_1$ radii of the same circle

then $\frac{CP_1}{CA} = \frac{CP}{CA}$

$$\therefore \quad \frac{BP_1}{AP_1} = \frac{BP}{AP}$$

thus for any point P_1 on this circle $\frac{BP}{AP}$ is a constant

and also

$$V_{P_1} = \frac{q}{2\pi\varepsilon} \ln \frac{BP_1}{AP_1} \qquad (14)$$

is a constant and the above circle is an equipotential for the system of line charges at A and B.

For this system we also have that

$$CP^2 = CA.\ CB \qquad (15)$$

A similar reasoning will establish similar equipotential of a circle of the same radius centre C' as shown in Fig. 7.

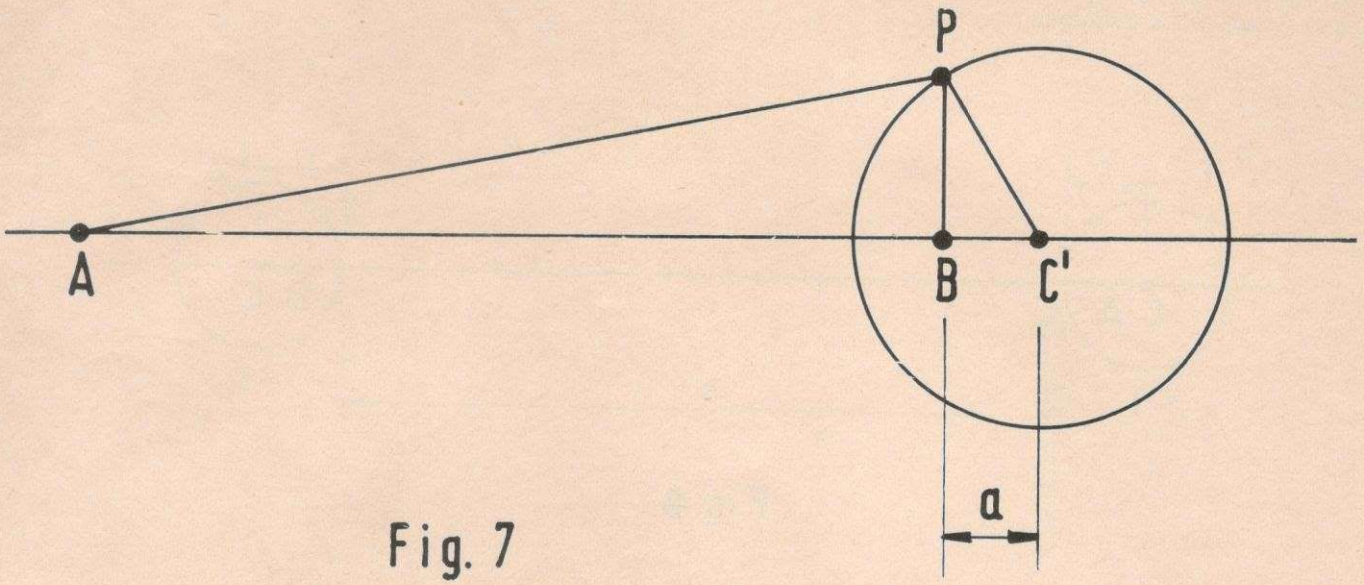


Fig. 7

The variable point P has the same potential on the circle in Fig. 6 and it is denoted by E, thus:

$$V_p = \frac{q}{2\pi\varepsilon} \ln \frac{BE}{AE} \qquad (16)$$

3. Potential at any point due to two line conductors of Radius r:

In Fig. 8, L_1 and L_2 are cylindrical conductors, centres C and C′ which are uniformly charged.

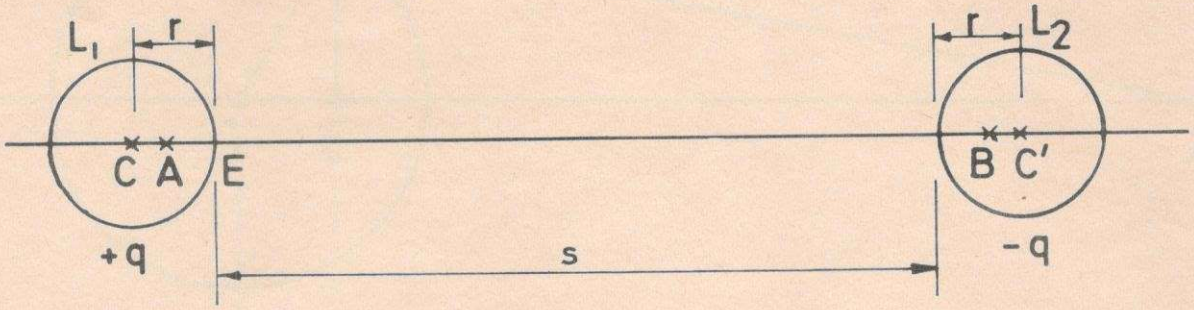


Fig. 8

It is easily seen from Fig. 6 where triangles CPB and CAP are similar that line charges at A and B are such that:

$$\frac{PB}{AP} = \frac{CB}{CP} \qquad (17)$$

For any point P line charges at A and B are equivalent to the cylindrical conductors:

$$\text{thus } CP^2 = CA \cdot CB \qquad (18)$$

It will be seen from Fig. 8 that

$$\begin{aligned} CC' &= s + 2r \\ CP &= r \\ CA &= a \\ C'A &= s + 2r - a \\ BE &= C'C - CE - BC' \\ &= s + 2r - r - a \\ &= s + r - a \\ AE &= CE - CA \\ C'B &= CE + s + C'P - a \\ &= 2r + s - a \end{aligned} \qquad (19)$$

and substituting these values in equation (18) above then

$$r^2 = a(s + 2r - a) \cdot \qquad (20)$$

$$\therefore \quad a^2 - a(s+2r) + r^2 = 0$$

$$\therefore \quad a = \frac{(s + 2r) - \sqrt{\{(s + 2r)^2 - 4r^2\}}}{2} \qquad (21)$$

$$\text{and} \quad 2a = s + 2r - \sqrt{(s^2 + 4rs)} \quad .. \quad .. \qquad (22)$$

The potential at P on the surface is:

$$V_p = \frac{q}{2\Pi\varepsilon} \ln \frac{BE}{AE} \qquad (23)$$

Using the values of BE and AE in equation (19) gives

$$BE = \frac{2s + 2r - s - 2r + \sqrt{(s^2 + 4rs)}}{2}$$

$$= \frac{s + \sqrt{(s^2 + 4rs)}}{2} \quad .. \quad .. \qquad (24)$$

$$\text{and AE} = \frac{2r - s - 2r + \sqrt{(s^2 + 4rs)}}{2}$$

$$= \frac{\sqrt{(s^2 - 4rs)} - s}{2} \quad .. \quad .. \qquad (25)$$

so that V_p becomes

$$\therefore \quad V_p = \frac{q}{2\Pi\varepsilon} \ln \frac{s + \sqrt{(s^2 + 4rs)}}{\sqrt{(s^2 + 4rs)} - s} \qquad (26)$$

Now Consider

$$\frac{s + \sqrt{(s^2 + 4rs)}}{\sqrt{(s^2 + 4rs)} - s} = \frac{\{s + \sqrt{(s^2 + 4rs)}\}^2}{s^2 + 4rs - s^2}$$

$$= \frac{s^2 + s^2 + 4rs + 2s\sqrt{(s^2 + 4rs)}}{4rs}$$

$$= \frac{s + 2r + \sqrt{(s^2 + 4rs)}}{2r}$$

$$\therefore \quad V_p = \frac{q}{2\Pi\varepsilon} \ln \frac{s + 2r + \sqrt{(s^2 + 4rs)}}{2r} \qquad (27)$$

4. Electric Stress along the line joining the centres of two current carrying conductors of equal Radius:

Let the conductors have radii r as shown in Fig. 9 and the spacing between them be s , then the following relations hold

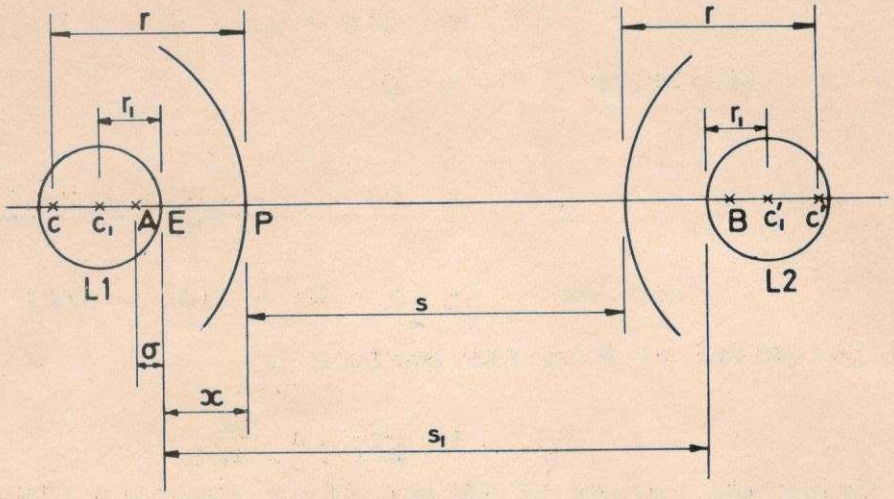


Fig. 9

$$AE = \frac{\sqrt{(s_1^2 + 4r_1 s_1)} - s_1}{2} \qquad (28)$$

$$V_p = \frac{q}{2\pi\varepsilon} \ln \frac{s + 2r + \sqrt{(s^2 + 4rs)}}{2r}$$

$$AP = \frac{\sqrt{(s^2 + 4rs)} - s}{2}$$

$$x = \frac{\sqrt{(s^2 + 4rs)} - s}{2} - AE$$

$$= \frac{\sqrt{(s^2 + 4rs)} - s}{2} - \sigma$$

$$2x = \sqrt{(s^2 + 4rs)} - 2\sigma - s$$

$$\therefore\ 2(x + \sigma) + s = \sqrt{(s^2 + 4rs)}$$

$$4(x + \sigma)^2 + 4s(x + \sigma) = s^2 - s^2 + 4rs$$

$$4(x + \sigma)^2 + 4s(x + \sigma) = 4rs$$

$$(x + \sigma)^2 + s(x + \sigma) = rs$$

$$\therefore\ s(r - x - \sigma) = (x + \sigma)^2$$

$$\therefore\ = \frac{(x + \sigma)^2}{r - x - \sigma} \qquad (29)$$

but $r - x - \sigma = CA$

$$= \frac{s + 2r - \sqrt{(s^2 + 4rs)}}{2}$$

$$\therefore 2r - 2x - 2\sigma = s + 2r - \sqrt{(s^2 + 4rs)}$$

$$\therefore -2x - 2\sigma = s - \sqrt{(s^2 + 4rs)}$$

Now $\sqrt{(s^2 + 4rs)} - s = 2x + 2\sigma$

and $V_p = \frac{q}{2\Pi\epsilon} \ln \frac{s + 2r + \sqrt{(s^2 + 4rs)}}{2r}$

$$= \frac{q}{2\Pi\epsilon} \ln \frac{2x + 2\sigma + 2r + 2s}{2r}$$

$$= \frac{q}{2\Pi\epsilon} \ln \frac{s + r + x + \sigma}{r}$$

$$= \frac{q}{2\Pi\epsilon} \ln \frac{\frac{(x + \sigma)^2}{r - x - \sigma} + r + x + \sigma}{r}$$

$$= \frac{q}{2\Pi\epsilon} \ln \frac{(x + \sigma)^2 + r^2 - (x + \sigma)^2}{r(r - x - \sigma)}$$

$$= \frac{q}{2\Pi\epsilon} \ln \frac{r}{r - (x + \sigma)} \quad . \quad . \quad . \quad (30)$$

Also $AB = CC' - 2CA = K$

Put $K = s + 2r - 2CA$

$$= s + 2r - (s + 2r) + \sqrt{(s^2 + 4rs)}$$

$$= \sqrt{(s^2 + 4rs)} \quad (31)$$

$$\therefore K^2 = s^2 + 4rs$$

$$= s(s + 4r)$$

$$= \{\frac{(x + \sigma)^2}{\{r - (x + \sigma)\}^2}\}\{4r + \frac{(x + \sigma)^2}{r - (x + \sigma)}\}$$

$$= \frac{(x + \sigma)^2}{\{r - (x + \sigma)\}} \{4r^2 - 4r(x + \sigma) + (x + \sigma)^2\}$$

$$= \frac{(x + \sigma)^2}{\{r - (x + \sigma)\}^2} \{2r - (x + \sigma)\}^2$$

$$\therefore K = (x + \sigma) \frac{2r - (x + \sigma)}{\{r - (x + \sigma)\}} \quad . \quad . \quad (32)$$

$$\therefore K\{r - (x + \sigma)\} = (x + \sigma)\{2r - (x + \sigma)\}$$

$$\therefore r\{K - 2(x + \sigma)\} = K(x + \sigma) - (x + \sigma)^2$$

$$\therefore r = \frac{(x + \sigma)\{K - (x + \sigma)\}}{K - 2(x + \sigma)} \quad . \quad . \quad (33)$$

thus $r - (x + \sigma)$ $= (x + \sigma)\left\{\frac{K - (x + \sigma)}{K - 2(x + \sigma)} - 1\right\}$

$= (x + \sigma)\ \frac{K - (x + \sigma) - K + 2(x + \sigma)}{(K - 2(x + \sigma)}$

$= \frac{(x + \sigma)}{K - 2(x + \sigma)}$

$\therefore \quad V_p = \frac{q}{2\Pi\epsilon} \ln \frac{(x + \sigma)\{K - (x + \sigma)\}}{K - 2(x + \sigma)} \cdot \frac{K - 2(x+\sigma)}{(x + \sigma)^2}$

$= \frac{q}{2\Pi\epsilon} \ln \frac{K - (x + \sigma)}{(x + \sigma)} \quad .. \quad .. \quad (34)$

$\frac{dV_p}{dx} = \frac{q}{2\Pi\epsilon} \cdot \frac{(x + \sigma)}{K - (x + \sigma)} \cdot \frac{-K}{(x + \sigma)^2}$

Since $\frac{d}{dx} \quad \frac{K - (x + \sigma)}{(x + \sigma)} = \frac{-(x + \sigma) - K - (x + \sigma)}{(x + \sigma)^2}$

$= \frac{- K}{(x + \sigma)^2}$

$\therefore \quad \frac{dV_p}{dx} = E_x = \frac{qK}{2\Pi\epsilon} \cdot \frac{1}{(x + \sigma)\{K - (x + \sigma)\}} \quad .. \quad (35)$

Now the maximum E occurs at the point x = 0

$\therefore \quad E \text{ max} = \frac{qK}{2\Pi\epsilon} \cdot \frac{1}{\sigma(K - \sigma)} \quad .. \quad .. \quad .. \quad (36)$

At the surface of the cylinder at point E the potential

$V_E = \frac{q}{2\Pi\epsilon} \ln \frac{K - \sigma}{\sigma}$

At point x = 0, at the other cylinder

$V = \frac{q}{2\Pi\epsilon} \ln \frac{K - \sigma}{\sigma}$

$\therefore \quad E \text{ mean} = \frac{V_E - V}{s_1}$

$= \frac{2q}{2\Pi\epsilon\ s_1} \ln \frac{K - \sigma}{\sigma}$

$= \frac{2q}{2\Pi\epsilon\ (K - 2\sigma)} \ln \frac{K - \sigma}{\sigma} \quad .. \quad .. \quad (37)$

field factor $\eta = \frac{E \text{ mean}}{E \text{ max}}$

$= \left\{\frac{2q}{2\Pi\epsilon\ (K - 2\sigma)} \ln \frac{K-\sigma}{\sigma}\right\} / \left\{\frac{q}{2\Pi\epsilon} \cdot \frac{K}{\sigma(K-\sigma)}\right\}$

$= \frac{2}{K - 2\sigma} \quad \frac{\sigma(K - \sigma)}{K} \ln \frac{K - \sigma}{\sigma}$

$$= \frac{2\sigma\ (K - \sigma)}{K(K - 2\sigma)} \ln \frac{K - \sigma}{\sigma} \quad .. \quad .. \quad (38)$$

the equation for η can be simplified further by putting $m = \frac{K}{\sigma} = \frac{2K}{K - s}$ (39)

thus $\eta = \frac{2\ (m - 1)}{m\ (m - 2)} \ln (m - 1) \quad .. \quad .. \quad (40)$

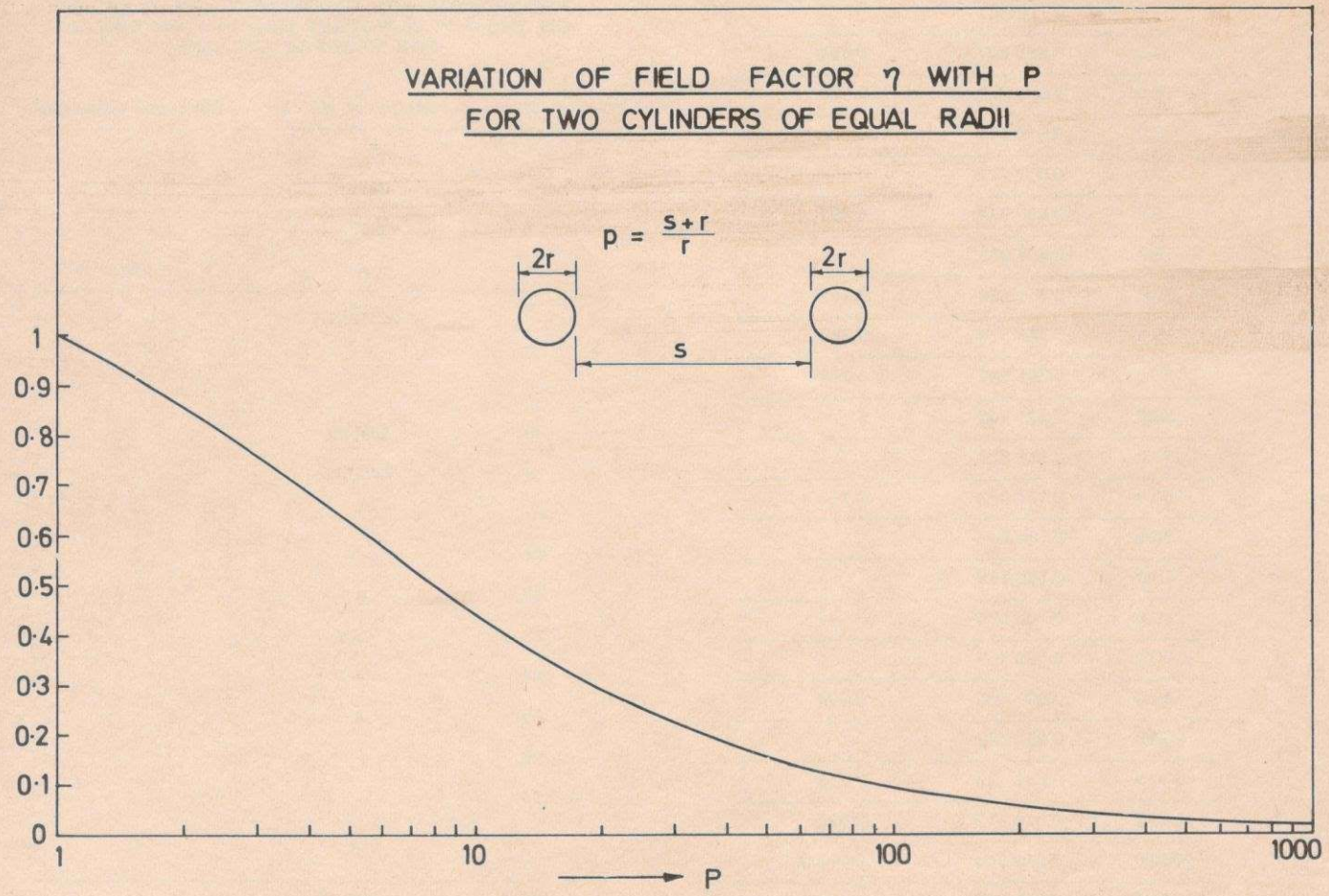


Fig. 10

TABLE 1

TABLE SHOWING SOME VALUES OF η AND p

$P = \frac{s+r}{r}$	η	*Value of as given in reference 4*
1.0	1.000 000	1.000
1.5	0.924 196	0.924
2.0	0.860 818	0.861
2.5	0.806 892	–
3.0	0.760 346	0.760
3.5	0.719 687	–
4.0	0.683 807	0.684
5.0	0.623 225	0.623
6.0	0.573 879	0.574
7.0	0.532 777	–
8.0	0.497 927	0.497
9.0	0.467 941	–
10.0	0.441 824	0.442
15.0	0.348 818	0.349
20.0	0.291 061	0.291
30.0	0.221 943	-
40.0	0.181 337	–
50.0	0.154 293	0.157 4
60.0	0.134 846	–
70.0	0.120 118	–
80.0	0.108 534	–
90.0	0.099 161	–
100.0	0.091 405	0.094
200.0	0.052 772	–
300.0	0.037 122	0.038
500.0	0.024 817	0.025
1000.0	0.013 304	0.013 8
1500.0	0.009 745	–

Conclusions

A simple expression has been derived for the field factor. This expression when compared with the expression derived by Schwaiger[1] 1919, and that of Dwight[2] 1924, make calculations quite easy. The object of the paper is to enable easy and fast calculations of field factors to be made. Thus it is easier to investigate simple systems of cylindrical electrodes under High Voltage for possible breakdown.

The tabulated values in Table 1 are compared with those given by Kind[4] 1970 using same method adopted by Prinz[3] 1969. The agreement between the values of the field factors in Table 1 shows that the expressions are the same although by careful choice of parameters the expression derived in this paper is very much simpler. The advent of pocket calculations with storage facilities ensures rapid calculations and the programme for Hewlett Parkard HP 65 pocket calculator is given in the Appendix 1 for reference. Appendix II shows a sample calculation, and figure 10 gives a plot of field factor/p variation.

APPENDIX I

PROGRAMME FOR CALCULATING FIELD FACTOR ON THE H.P. 65 POCKET CALCULATOR SUITABLE FOR STORAGE ON TAPE

Step	*Function on HP 65*	*Expression Calculated*
1.	f	
2.	PRGM	
3.	LBL	
4.	A	CLEAR A
5.	ENTER	
6.	1	
7.		
8.	ENTER	
9	ENTER	
10.	f^{-1}	
11.	$\sqrt{x}$	
12.	g	
13.	$x \gtrless y$	
14.	4	
15.	X	
16.	+	
17.	f	
18.	$\sqrt{x}$	K/r
19.	ENTER	
20.	ENTER	
21.	g	
22.	R_Λ	
23.	–	$2\sigma/r$
24.	$\frac{\cdot}{\cdot}$	
25.	2	
26.	X	m

Step	Function on HP 65	Expression Calculated
27.	ENTER	
28.	ENTER	
29.	ENTER	
30.	2	
31.	–	
32.	X	
33.	g	
34.	1/x	
35.	g	
36.	X≷y	
37.	1	
38.	–	
39.	ENTER	
40.	f	
41.	LN	
42.	X	
43.	X	
44.	2	
45.	X	$\frac{2(m-1)}{m(m-2)}$
46.	DSP	
47.	6	
48.	RTN	END OF PROGRAMME

APPENDIX II

Example: To find the field factor when p = 3

Solution:

From table 1

$$p = \frac{s}{r} + 1$$

$$\text{and } K = \sqrt{(s^2+4sr)} = r\sqrt{\left(\frac{s^2}{r^2} + 4\frac{s}{r}\right)} \quad (31)$$

From Fig.10 we can write

$$\sigma = \frac{K - s}{2}$$

$$\text{Also } m = \frac{K}{\sigma} = \frac{2K}{K-s} \quad (39)$$

$$\text{and } \eta = \frac{2(m-1)}{m(m-2)} \ln(m-1) \quad (40)$$

Using $p = 3$

then $\frac{s}{r} = 2$

$$K = r\sqrt{(4+8)} = 3.464102\ r$$

$$m = \frac{2K}{K-s} = \frac{2 \times 3.464102}{3.464102 - 2}$$

$$= 4.732051$$

$$\frac{2(m-1)}{m(m-2)} = \frac{2 \times 3.732051}{4.732051 \times 2.732051}$$

$$= 0.577350$$

Also $\ln (m-1) = \ln 3.732051$

$$= 1.315958$$

$$\therefore \eta = \frac{2(m-1)}{m(m-2)} \ln (m-1)$$

$$= 0.57735 \times 1.316958$$

$$= 0.760346$$

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