STRESS COEFFICIENT METHOD FOR THE DETERMINATION OF BENDING STRESSES IN THICK CURVED BARS

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ABSTRACT

In the case of pure bending, equilibrium requires that, the sum of the normal forces distributed over the cross section must be zero and the moment of these forces equal to the external moment. These equilibrium conditions are used to determine coefficients that describe the stress distribution over the cross section of a curved bar. The method requires only the calculation of the stress coefficients.

Numerical examples are presented in order to demonstrate the validity of the method and the possibilities of application.

1. INTRODUCTION

The simple theory of bending of a straight beam is employed to determine bending stresses in a curved bar if the radius of curvature is large compared with the dimensions of the cross section. When the radius of curvature is of the same order as the dimensions of the cross section it is no longer possible to apply the straight beam theory. Due to bending the neutral axis does not pass through the centroid and stress is not proportional to distance from the neutral axis.

Assuming that there is no lateral pressure between the longitudinal fibers, the bending stress (1) at the distance y from the centroidal axis and normal to the cross section is given as

$$\sigma = \frac{E(y-e)\ \Delta d\theta}{(R-Y)\ d\theta} \qquad (1)$$

where R is the radius of curvature, Eq (1) contains two unknowns, the distance e of the neutral axis from the centroid and the angle of ratation $\Delta d\theta$ of adjacent cross sections with respect to each other and about the neutral axis.

In the case of pure bending the sum of the normal forces distributed over the cross section is zero and the moment of these forces is equal to the external moment. The determination of the two unknowns in Eq (1), therefore, requires the use of the following two equations (2) of statics:

$$\int_A \sigma dA = \frac{EAd\theta}{d\theta} \int_A \frac{(y-e)\ dA}{(R} = 0 \qquad (2)$$

$$\int_A \sigma y dA \quad \frac{EAd\theta}{d\theta} \int_A \frac{(y-e)\ ydA}{(R-y)} = M \qquad (3)$$

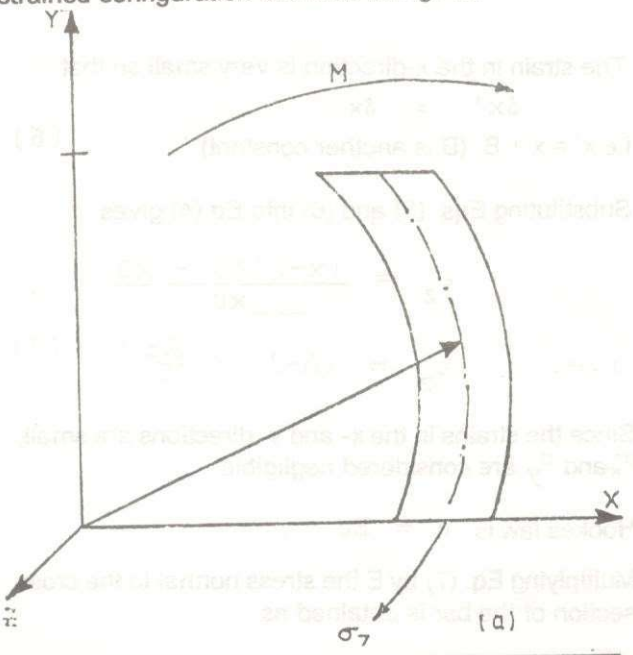
These equations may be transormed (3) to determine Ädθ and e for any particular case of a curved bar. The determination of Ädθ and e is very tedious.

The method presented in this paper alleviates this difficulty.

2. STRESS COEFFICIENTS

Equation (1) defines the bending stress, normal to the cross-section, at a distance y from the centroidal axis. The equilibrium conditions, Eqs. (2) and (3), used for the determination of Ädθ \$ e, can be used to determine coefficients that describe the stress distribution in a thick curved bar.

Figure 1a shows an element of a curved bar at a distance x from the centre of curvature. The strained configuration is shown in Fig. 1b.



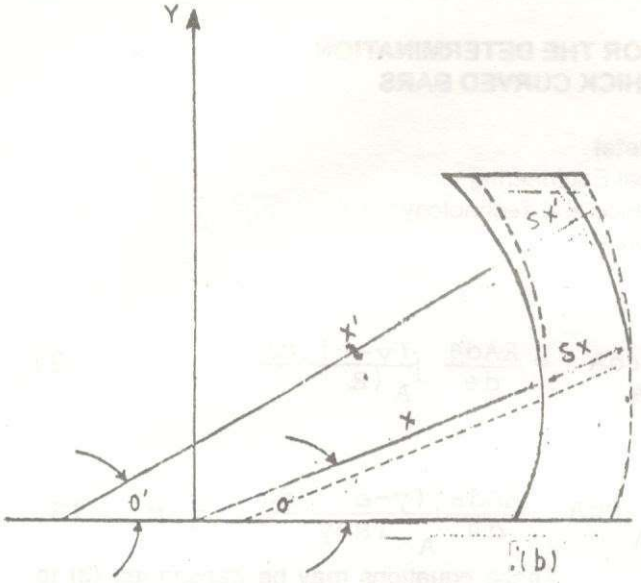


Fig. 1 Curved bar under bending load

The angle è is small so that the strain in the z-direction is

$$\varepsilon_z = \frac{x'0' - x0}{x0} \qquad (4)$$

Assuming that the entire transverse section of the bar originally plane remains plane and normal to the longitudinal fibers of the beam after bending, then

$$\frac{d\theta}{dx} = \frac{d\theta'}{dx'} = \theta \qquad (5)$$

$$\text{or} \quad \frac{\theta'}{\theta} = A \text{ (constant)}$$

The strain in the x-direction is very small so that

$$\delta x' \simeq \delta x$$

i.e $x' = x + B$ (B is another constant) (6)

Substituting Eqs (5) and (6) into Eq (4) gives

$$\varepsilon_z = \frac{(x-B)AO - x0}{x0}$$

$$\text{i.e.} \quad \varepsilon_z = (A-1) + \frac{AB}{x} \qquad (7)$$

Since the strains in the x- and y- directions are small, σ_x and σ_y are considered negligible.

Hookes law is $\sigma = E\varepsilon$

Multiplying Eq. (7) by E the stress normal to the cross section of the bar is obtained as

$$\sigma_z = E(A-1) + \frac{E(AB)}{x} \qquad (8)$$

Eq (8) may be written as

$$\sigma_z = \alpha + \frac{\beta}{x} \qquad (9)$$

Where $\alpha = E(A-1)$ and $\beta = E(AB)$.

The coefficients and are determined from the equilibrium conditions, and Eqs. (2) and (3) can be written as

$$\int \sigma_z dA = \sum(\alpha + \frac{\beta}{x})\delta A = 0 \qquad (10)$$

$$\int \sigma_z x dA = \sum(\alpha + \frac{\beta}{x}) x \delta A = M \qquad (11)$$

Equations (10) and (11) are the Equilibrium conditions in terms of the stress coefficients α and β It can be observed that the coefficients α and β are a function of the modulus of elasticity of the bar material. For a given curved bar, therefore, irrespective of the material of the bar, it is only necessary to calculate the area of the section and the first moment of that area of the section and the first moment of that area to determine the stress coefficients and so the stress distribution. A few solved problems will demonstrate the validity of the method.

3. ILLUSTRATIVE EXAMPLES

(i) Figure 2 shows a curved bar of rectangular section subjected to bending moment of 1.2kNm. It is required to evaluate the greatest and least bending stresses.

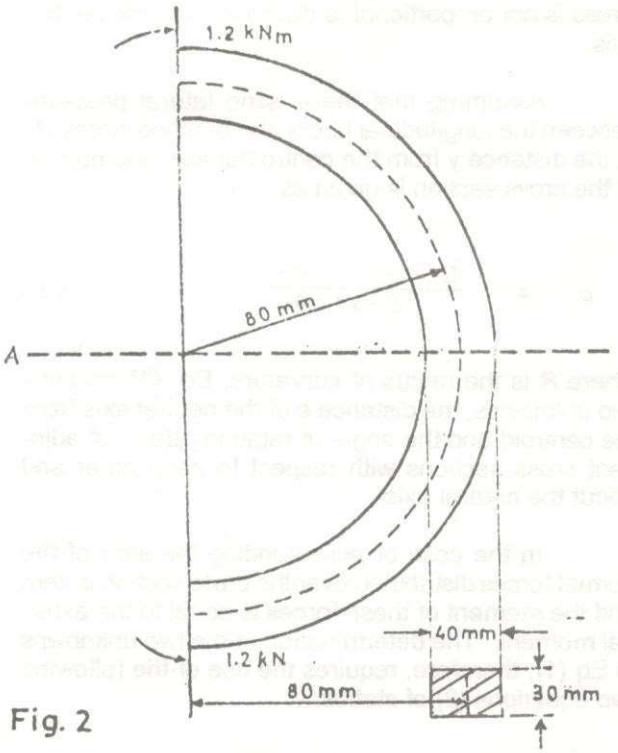


Fig. 2

The stress acting on a section through AB at a distance x from the centre of curvature is given by Eq. (9) as

$$\sigma = \alpha + \frac{\beta}{x} \qquad (a)$$

From equilibrium considerations;

$$\sum F = 0 = \sum(\alpha + \frac{\beta}{x})\delta A \qquad (b)$$

$$\sum M = 0 = \sum(\alpha + \frac{\beta}{x}) \times \delta A - 12\times10^{6}\ NM \qquad (c)$$

Eqs. (b) and (c) may be written as

$$\alpha\sum\delta A + \beta\sum\frac{\delta A}{x} = 0 \qquad (d)$$

$$\alpha\sum\times\delta A + \beta\sum\delta A = -12\times10^{6} \qquad (e)$$

From Fig. 2

$$\sum\delta A = 40\times30mm^2 = 1200mm^2$$

$$\sum\times\delta A = 80(40\times30)mm^3 = 96000mm^3$$

$$\sum\frac{\delta A}{x} = \int_{60}^{100}\frac{bdx}{x} = 30\int_{60}^{100}\frac{dx}{x} = 30\ In\ \frac{5}{3}$$

$$= 15.3245mm$$

Substituting values of δA, $x\delta A$ and $\frac{\delta A}{x}$ into Eqs. (d) and (e) gives

$$1200\alpha + 15.3246\beta = 0 \qquad (f)$$

$$96000\alpha + 1200\beta = -1.2\times10^{6} \qquad (g)$$

$$\therefore \alpha = -\frac{15.3246}{1200}\beta = -0.01277\beta \qquad (h)$$

Substituting Eq. (h) into Eq. (g) gives

$$\beta = 46296.2$$

and from Eq. (h)

$$\alpha = -591.1$$

At the convex side of the bar, x = 60mm

$$\hat{\sigma} = -591.1 + \frac{46296.2}{60} = 180.4\ \frac{N}{mm^2}$$

At the concave side of the bar, x = 100mm

$$\therefore \quad \overset{v}{\sigma} = -591.1 + \frac{46296.2}{100} = 128.23\ \frac{N}{mm^2}$$

ii) The crane hook shown in Fig .3 was analyzed by Timoshenko (I) using the Winker's approximate theory and obtained the maximum stresses as 56.54 N/mm² and 33.1 N/mm² respectively. As a check of the method presented in this paper we also analyse this problem.

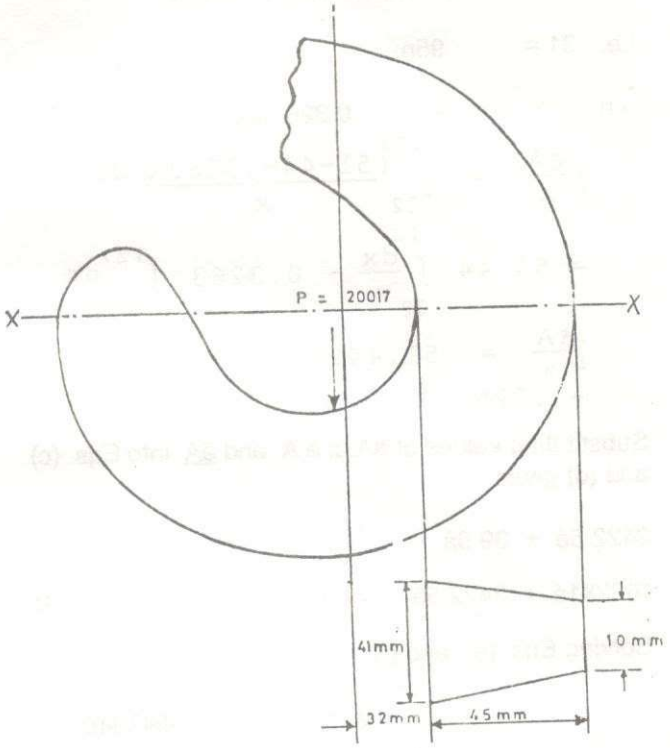


Fig. 3

The stress acting on section xx is

$$\sigma + \alpha + \frac{\beta}{x}$$

Equilibrium requires that

$$\sum F = 0 = \sum(\alpha + \frac{\beta}{x})\ \delta A - P$$

$$\sum M = 0 = \sum(\alpha + \frac{\beta}{x}) \times\delta A$$

Rewriting Eqs. (a) and (b)

$$\alpha\sum\delta A + \beta\sum\frac{\delta A}{x} = 20017\ N \qquad (c)$$

$$\alpha\sum\times\delta A + \beta\sum\frac{\delta A}{x} = 0 \qquad (d)$$

From Fig. 3

$$\sum\delta A = (95\times10) + 2(\tfrac{1}{2}\times15.5\times95)$$

$$= 2422.5mm^2$$

$$\sum\times\delta A = 79.5\ (95\times10) +$$

$$63.7\ 2(\tfrac{1}{2}\times15.5\times95 = 169291mm^3$$

$$\sum\frac{\delta A}{X} = \int_{32}^{127}\frac{\delta A}{x}$$

Let b = m-nx

From Fig. 3 when b = 41, x = 32, ∴41 = m-32n

When b= 10, x, = 127 ∴10 = m - 127n

i.e. 31 = 95n;

∴n = 0.326 and m = 51.44

$$\sum \frac{\delta A}{x} = \int_{32}^{127} \frac{(51-44-.3263x)dx}{x}$$

$$= 51.44 \int_{32}^{127} \frac{dx}{x} - 0.3263 \int_{32}^{127} dx$$

$$\sum \frac{\delta A}{x} = 51.44 \ln \frac{127}{32} - .326(127 - 32) = 39.9$$

Substituting values of äA, x ä A and äA into Eqs. (c) and (d) gives

2422.5á + 39.9â = 20017 (e)

169281á + 2422.5â = 0 (f)

Solving Eqs (e) and (f);

á = -547346

â = 3824.85

The Stress distribution is then

$$\overset{v}{\sigma} = -54.8346 \frac{3824.85}{124} = -24.6 \frac{N}{mm^2}$$

$$\hat{\sigma} = -54.7346 \frac{3824.85}{32} = 67.79 \frac{N}{mm^2}$$

The percentage deference compared with Timoshenko's value are

$\overset{v}{\sigma}$ = -2.56%, and σ = + 1.27%.

iii) **Fig. 4** Shows a thick curved beam of a rectangular cross-section loaded wires as shown. The area of each wire determined, given that the greatest stress in the beam is equal to half the stress in each wire.

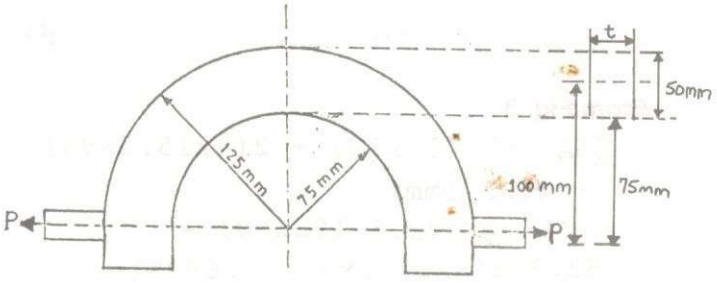


Fig 4.

This problem was solved by Richards (5) and obtained the thickness t as 18.75mm. For the purpose of comparison we solve this problem by the stress coefficient method.

Stress in wire, $\sigma_w = \frac{P}{30mm^2}$

$\therefore P = 30 \sigma_w$

For equilibrium

$\sum F = 0 = \alpha(\sum d\Lambda) + \beta(\sum \frac{A}{x}) - P$

$\sum M = 0 = \alpha(\sum \delta Ax) + \beta(\sum \delta A)$

$\sum \delta A = 50\ t$

$\sum \delta Ax = 50\ t \times 100 = 5000\ t$

$\sum \frac{\delta A}{x} = \int_{75}^{125} \frac{bdx}{x}$, where b = t

$\sum \frac{\delta A}{x} = t \int_{75}^{125} \frac{dx}{x} = t \ln(\frac{125}{75}) = 0.51\ t$

From Eq. (b)

$\beta = -100\alpha$ (c)

Substituting for β in (a) gives

$\alpha = -\frac{P}{t}$ (d)

Combining Esq. (c) and (d)

$\beta = 100 \frac{P}{t}$

Now

$\sigma = \alpha + \frac{\beta}{x}$

$\sigma = -\frac{P}{t} + \frac{100\ P}{75\ t} = 0.333 \frac{P}{t}$

But

$\sigma = \frac{1}{2}\sigma_w = \frac{P}{60} = \frac{P}{60}$

$\therefore \hat{\sigma} = \frac{P}{60} = 0.333 \frac{P}{t}$

$t = 0.333 \times 60 = 19.98mm$

The percentage difference compared with Richard's value of 18.75mm is 6.2%

discussion

Equation (I) was developed (I) in 1867. It is an approximate e and Ädè. For a rectangular cross-section bar in particular e is very small compared with the mean radius of curvature. The expression for e has a log term and it is not easy to calculate with sufficient accuracy. It becomes necessary, therefore, to expand the log term into a convergent and truncate it for the desired accuracy.

Since Eq. (I) itself is an approximate theory, the stresses determined are also approximate value. The degree of accuracy will, ofcourse, depend on how accurately e and Ädè are calculated. In the face of these difficulties and approximations, a shift to simpler method for the determination of stress distribution in a thick bar will be advantageous.

4. CONCLUSION

This paper presents a method that permits the determination of stress distribution in a thick curved bar in a rather easy way. unlike the Winkler's approximate theory, which requires the transformation of integral equations of forces and moments for the determination of the distance of the neutral axis from the centroid and the angle of rotation of adjacent only the calculation of the cross-section, the method presented in this paper requires only calculation of the cross-sectional area of the bar for the determination of stress coefficients that describe the stress distribution in the bar.

5. REFERENCES

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