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FLEXURAL BUCKLING ANALYSIS OF CRACKED BEAM-COLUMNS BY THE FINITE ELEMENT METHOD

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ABSTRACT

The crack induced flexibility and hence crack-spring stiffness relevant for the study of pre-cracked concentric and eccentrically loaded columns with different boundary conditions is derived asymptotically herein. The finite element procedure is then employed to study the column response and sensitivity to both concentric and eccentric loads. As expected the results obtained show a remarkable response sensitivity and reduction in strength with increasing crack depths and are compared with those published in the literature. The results also show that influence of mid height crack induced flexibility is to reduce the load carrying capacity of such columns provided they are loaded eccentrically.

NOTATION

A	surface area of crack
B	Stiffness of cracked component
b	Width of component plate
c	crack length
E	Youngs modulus
G	Energy release rate
F	Flexibility (compliance)
K	Non-dimensional Stress intensity factor
$K_{I=}$	Stress intensity factor
K	Stiffness
L	Length
M	Applied moment
q	Nodal displacement
u	Axial displacement
v	Transverse displacements
υ	Poissons ratio
λ	Load level

1. INTRODUCTION

Many researchers have extensively studied the stability of slender columns, following the pioneering work by Euler[10]; leading to the famous Euler critical buckling loads for axially loaded columns with different fixity conditions.

Another aspect of the study of axially loaded columns has been concerned with the response of columns with geometric imperfections to applied loads. Again following the pioneering work by Koiter[1], leading to the famous ½ and 2/3 power law equations, there followed numerous analytical [1-3], experimental [4-5] and numerical [6-10] publications on the stability of geometrically imperfect columns.

On the other hand, experimental works on the behavior of cracked concentric and eccentrically loaded columns have been reported by [4, 5]. The findings by references [4] and [5] reveal that the carrying capacity of axially loaded columns is not affected by edge cracks except if the column is loaded eccentrically. This viewpoint was also confirmed in later theoretical study by [5].

It is well known that columns, beam-columns and tie beams are used in very important Engineering Structures such as bridges, offshore structures, ship structures and aircraft structures where the presence of cracks will lead to either premature buckling or failure by fracture. This explains the interest and determination of the above researchers in the study of cracked columns as reviewed above.

However, with recent developments in computer technology (cheaper micros are now available) coupled with scarce materials, there is still need to explore the use of the new technological tool (the computer) in assessing previously obtained technological findings for more efficient use of scarce materials in designs to accommodate the demand for dwindling materials [11, 12]. There is also need to study the behavior of cracked thin walled structures (columns) with consideration of different boundary conditions and loading. Unfortunately, as can be seen from the above review, the aspect of the use of the computer for the study of load carrying capacity of thin-walled cracked or notched columns is scarce. Thus the present work seeks to fill the scarcity.

2. ASSUMPTIONS

The following assumptions are made in the present study:

I. The column is prismatic and thin-walled.
II. Plane sections remain plane after buckling
III. Only doubly symmetric sections are considered
IV. The column is pre-cracked
V. St. Venant Torsion is not considered
VI. Warping Torsion is not considered

3. CRACKED COLUMN MODEL

For a flexural torsional buckling analysis we assume the following structural interaction model based on the fact that the energy release rate due to bending is greater than that due to axial compression. Based on this assumption we have:

$$G = \tfrac{1}{2} M^2 \, dF/dA \qquad (1)$$

Assuming plane strain condition we also have:

$$G = (1-\nu^2) K_1^2 / E \qquad (2)$$

In which K_1 is the model stress intensity factor

Therefore

$$dF/dA = 2(1-\nu^2) K_1^2 / EM^2 \qquad (3)$$

From [3], see also [10]

$$K_1 = 6MK_0^2 / a\left(c/ab\right)^{1/2} \qquad (4)$$

For a thin strip of column loaded with a bending moment M and having crack depth c and assuming in he case of a unit width of crack that a surface crack of area A=c.1=c; has taken place, then we have

$$dF/dA = 2(1-\nu^2) K_1^2 / EM^2 \qquad (5)$$

From figure 2 for a strip with a single crack c on one side, the dimension a is given as:

$$a = b = c \qquad (6)$$

Then from equation (4) we have

$$K_1 = 6MK_0 / a\left[1/b\left(c/b - 1\right)\right]^{1/2} \qquad (7)$$

When equation (7) is squared we have:

$$K_1^2 = 36M^2 K_0^2 / a^2 b(c/b - 1) \qquad (8)$$

Therefore substitution of equation (8) into equation (5) gives:

$$dF/dc = 72(1-\nu^2) / Eba^2\left(c/b - 1\right)K_0^2 \qquad (9)$$

Integrating equation (9) we have:

$$F = 72(1-\nu^2) / Eba^2 \int_c (c/b - 1) K_0^2 c \, dc \qquad (10)$$

Again from [3] we have:

$$K_0 = 1.122\left[1 - 2.717(c/b) + 0.523(c/b)^2\right] \quad (11)$$

Equation (11) is squared and substituted into equation (10) and after integration gives:

$$F=72\left(1-v^2\right)/Eba^2 f(\alpha) \quad (12)$$

In which

$$f(\alpha) = [-1.258884\ (c/b) + 4.0498\ (c/b)^2 - 2.6339\ (c/b)^3 + 1.15968\ (c/b)^4 -$$

$$- 0.7845(c/b)^5 + 0.0579(c/b)^6 \quad (13)$$

The relationship between local compliance and local stiffness is:

$$K=1/F \quad (14)$$

Substituting equation (12) into equation (14) we have the local stiffness K as:

$$K = Eba^2/72\ (1-v^2)\ f(a) \quad (15)$$

$$B = K\ x\ L\ in\ (knm^2) \quad (16)$$

In which L is the length of the beam or column. The Euler buckling load for un-cracked column is given as:

$$P_{cr} = \lambda\pi^2\ EI/L^2 \quad (17)$$

In which λ is load level and takes the value of unity in equation (17) for a pin-pin column and EI is the stiffness of un-cracked column. For the cracked column, equation (17)

Becomes:

$$P_{cr} = \lambda\pi^2\ B/L^2 \quad (18)$$

As can be seen from equations (17) and (18), the accuracy of a given numerical solution depends on the value of λ obtained. The closer to unity the value of λ, the better the solution for the pin-pin boundary conditions, this is similar for other boundary conditions. The accuracy of the program developed for the present work is tested with different values of λ returned by other researchers for various boundary conditions and examples.

In the next section, the finite element procedure is presented and the results from the finite element computer program for the cracked column is calculated and compared with cracked columns using theoretical and experimental findings published by others.

4. FINITE ELEMENT MODEL

In order to carry out a linear stability analysis by the finite element method, it is usual to cast the stability problem in the form

$$[K]+\lambda[G]\{X\}=0 \quad (19)$$

In which

[K] is the assembled stiffness matrix

[G] is the assembled stability or geometric stiffness matrix

λ is load level to be calculated

{X} is a vector of global nodal displacements.

4.1 ELEMENT STIFFNESS MATRICES

The elastic stiffness and stability matrices for the element shown in figure 2 are derived from the strain energy of the element which is given as:

$$U = \int_{vol} \varepsilon^T D \varepsilon \, dvol \qquad (20)$$

In which D is the material property matrix=E in the present formulation.

The component of strain in the x-direction is given as [14]:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - y\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \qquad (21)$$

In terms of the material property E, the strain energy is given as:

$$U = \frac{E}{2}\int_{vol} \varepsilon_{xx}^{\ 2} dvol \qquad (22)$$

The displacement field of the element shown in figure 2 is given as [14]:

$$u = (1 - \frac{x}{l})q_1 + (\frac{x}{l})q_4 \qquad (23)$$

$$v = (1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3})q_2 + (\frac{2x^2}{l^2} + \frac{x^3}{l^2})q_3 + (\frac{3x^2}{l^2} - \frac{2x^3}{l^3})q_5 + (-\frac{x^2}{l} + \frac{x^3}{l^2})q_6 \qquad (24)$$

In a compact notation, the displacement fields' u and v are given as:

$$\left\{\frac{u}{v}\right\} = \left[\begin{matrix} \phi_1 & 0 & 0 & \phi_4 & 0 & 0 \\ 0 & \phi_2 & \phi_3 & 0 & \phi_5 & \phi_6 \end{matrix}\right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \qquad (25)$$

In which $\phi_i (i = 1,2 - - - - 6)$ are the coefficients of the nodal displacements q_i (i=1,2-------6) as shown in equations (23) and (24). The displacements u and v are partially differentiated with respect to x and are substituted into equation (22) to give the cracked strain energy of the element as:

$$U = EA/2L(q_1^2 - 2q_1q_4 + q_4^2) + 2B/L^3(3q_2^2 + q_3^2 + 3q_5^2 + q_6^2$$

$$+ 3q_2q_5 + 3q_2q_6 - 3q_3q_5 + q_3q_6 - 3q_5q_6) + EA/L^2(q_4^2 - q_1^2)$$

$$(3/5q_2^2 + 1/15q_3^2 + 3/5q_5^2 + 1/6q_6^2 + 1/10q_2q_3 - 6/5q_2q_5 + 1/10q_2q_6$$

$$-1/10q_3q_5 - 1/30q_3q_6 - 1/10q_5q_6) \qquad (27)$$

Taking

$1/L(q_4 - q_1) = strain = \varepsilon \ and \ (q_1 - q_4) = constant$ and noting that Castigliano's theorem is applicable here gives:

$$\{p\} = \delta U / \delta q_i (i = 1 - - - - - 6) = [k_T]\{q\} \qquad (28)$$

In which

$[k_T]$ is tangent stiffness matrix given as

$$[k_T] = [k_e] + \lambda[g] \qquad (29)$$

Both $[k_e]$ and $[g]$ are listed in Jiki [10]. {q} is the displacement vector as has been defined above in equation (25). λ is a load level. Equation (29) is assembled into system equations as:

$$\{P\} = [[K] + \lambda[G]]\{Q\} \qquad (30)$$

In which

{P} is assembled load vector

[K] is assembled elastic stiffness matrix

$$U = \frac{EA}{2}\int_L (\frac{\partial u}{\partial x})^2 dx + \frac{B}{2}\int_L (\frac{\partial^2 v}{\partial x})^2 dx + \frac{EA}{2}\int_L \frac{\partial u}{\partial x}(\frac{\partial v}{\partial x})$$

Equations (23) and (24) are partially differentiated and substituted into equations (26) and carrying out appropriate integrations, the strain energy of equation (26) is given as:

[G] is assembled geometric stiffness matrix

Equation (30) is an eigenvalue equation and is solved by the generalized Jacobi algorithm [9].

5. EXAMPLES FOR COMPARISON STUDIES

The first example used to show the accuracy of the present formulation and the software code used for the extraction of the eigenvalues based on procedures from [9] is taken from [7]. In the example, the stability of a fixed-pin column shown in figure 3 is investigated. Using two elements and after expansion of the determinant, [7] obtained the lowest root of the resulting equation $\mu = \lambda L^2 / EI = 5.1772$. note that for un-cracked column the stiffness B=EI. The present investigator used the same two elements with fixed-pin boundary conditions and the program written in

Wateroo Fortran WatFor 77, using the generalized Jacobi algorithm as detailed by [9], a value of $\mu = 5.17720$ was obtained. Table 1 contains details of the results; together with percentage errors for a two element discretization. The second example is taken from [8]. Again using a two element discretization. [8] solved for the lowest root of the determinant equation and obtained the value for the load parameter as:

$\lambda = 2.48$, in which $\lambda = pl / EI$. this gives $P_{cr} = 2.48 EI / L^2$ or

$P_{cr} = 9.92 EI / L^2 = k\pi^2 EI / L^2 = 1.005\pi^2 EI / L^2$. in this case the factor k=1.005. for the same pin-pin column model, the writer's program gives a value of $\lambda = 2.48596$. This gives $P_{cr} = 2.48596 EI / L^2$ or $9.94384\, EI / L^2 = 1.0075\pi^2 EI / L^2$. therefore k=1.0075.

The third example employed a fixed-fixed column model (see figure 3) to check accuracy of the software code developed. The program outputs the critical value of $\lambda = 10 \; or \; 4\lambda = 40$, therefore k=4.0529. The fourth example, also shown in figure 3 employed a cantilever column model for the study. Again using two elements the computer program outputs the critical value of $\lambda = 0.61717$. Therefore the critical buckling load

$P_{cr} = 0.62 EI / L^2 \; or \; P_{cr} = 9.92 EI / 4L^2$. In a standard form we have $P_{cr} = k\pi^2 EI / 4L^2 = 1.005\pi^2 EI / 4L^2$. in this case k=1.005. Table 1 compares results obtained for present work with standard works of Euler[10], Hartz[8] and Prezieniky[7].

The fifth example for comparison studies is taken from [4] and is shown in figure 4. It consists of experimental tests on $12.5\text{mm} \times 12.5mm \times 300mm$ long rolled bars made of 7075-T6511 aluminium. The columns containing cracks had notches put in at the center of specimens either on one side or on both sides. One of such specimens was selected, tested and the failure loads for seven crack depths at five different eccentricities were recorded.

Using the asymptotic expression presented in equation (18) together with buckling parameter λ, the critical loads were obtained for each of the five eccentricities using the finite element method. The results of some eccentrically loaded cracked columns reported earlier by Liebowitz and co-workers[4], are compared with the finite element analysis carried out in the present work is shown in figure 4. Agreement between results of the present work and those of [4] is good. Close observation of figure 4 however, one notices that there were slight discrepancies of about 3-5 percent for small notch depths between the results of [4] and the present work, but as the notch depths get larger, or deeper, the agreement between [4] and the present work become closer. The sixth example is the last example which studies the response sensitivity of a given column to crack. Obviously a crack or notch represents loss of material which cannot enhance strength but for concentrically loaded columns, Liebowitz and co-workers [4], [5] have found that the strengths of such columns remain the same with those for un-cracked ones. This is because the crack surfaces bear the load.

To study the effect of an edge crack on flexibility of the cracked column, a 300mm×600mm×50mm cantilever column with tip axial and transverse loads as shown on figure 5 was discretised for a plane stress analysis. A constant strain triangular finite element was used for the course mesh shown on the figure. A program developed by Jiki[10] using the Wateroo Fortran (WatFor 77) was used to obtain results. The response of a particular node, that is node 7, was used for the study. The tip transverse load W was kept constant at 20KN while the axial load was incremented from 25KN to 200KN with increment step of 25KN per step. Table 2 shows the deflection response of node 7 and figure 6 shows the graph of load versus deflection response for the same node. It was found that for zero crack, the hogging deflection of node 7 was slow and increased progressively up to axial load of 200KN. However, when the crack was introduced there was a reversal of deflection of the node from positive values up to a load of 75KN, to negative values from loads of 100KN and up to 200KN. This shows that only a 10mm edge crack reduces the stiffness of the column appreciably. However more work is required to understand fully the issue of crack sensitivity to flexibility.

This point may form the subject for further study.

6. CONCLUSION

Following the results obtained for the present study, the following conclusions have been reached.

A crack or a notch introduces a loss of material which reduces the stiffness of the member and which leads to a reduced load carrying capacity of the column.

Crack surfaces have been found to assist in load sustenance particularly for concentrically loaded columns. However, the effect of local buckling has not been examined even for concentrically loaded cracked or notched columns. Failure of eccentrically loaded columns and beam-columns maybe by loss of stiffness and buckling, however, deeper fatigue cracks may lead to pre-mature failure by fracture.

From the above evidence about cracked thin walled-columns and beam-columns. It is concluded that cracked components in sensitive structures such as ships, offshore, air craft and space vehicles should be replaced with un-cracked ones immediately when the cracks are detected.

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APPENDIX 1

Table 1: **Comparison of critical load factors for axially loaded columns**

Column model	Euler [10]	Hartz[8]	Present
Factor	k	k	k
Pin-Pin	1.0	1.005 (0.5%)	1.0075 (0.75%)
Fixed-Fixed	4.0	4.052 (1.3%)	4.052 (1.3%)
Fixed-Free	1.0	1.055 (5.5%)	1.005 (0.5%)
Fixed-Pin	$\pi^2/(0.699\text{L})^2$		$\pi^2(0.724L)^2$ (2.5%)

Table 2: **Cantilever column model with point load of 20KN at Tip for displacement sensitivity analysis of node 7 to crack for element 3 for discretization is shown in figure 4b.**

Crack depths 0 10mm 20mm

Load (KN)	$\delta_0 \times 10^{-2}$ **mm**	$\delta_{10} \times 10^{-2}$ **mm**	$\delta_{20} \times 10^{-2}$ **mm**	$(\delta_{10} - \delta_0) \times 10^{-2}$ **mm**	$(\delta_{20} - \delta_0) \times$ **mm**
(1)	(2)	(3)	(4)	(5)	(6)
0.0	0.0	0.0	0.0	0.0	0.0
25.00	3.515	3.906	3.940	0.391	0.425
50.00	3.624	3.832	3.867	0.208	0.243
75.00	3.733	3.759	3.794	0.026	0.061
100.00	3.842	3.685	3.722	0.157	0.120
125.00	3.991	3.611	3.649	0.340	0.302
150.00	4.060	3.537	3.576	0.523	0.484
175.00	4.169	3.464	3.504	0.705	0.665
200.00	4.278	3.390	3.431	0.888	0.847

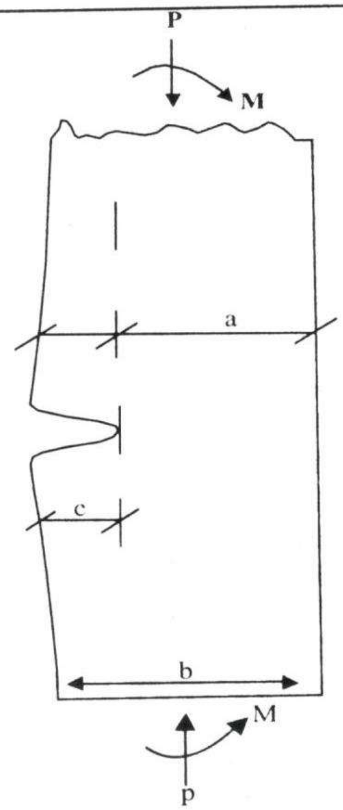


Figure 1 Edge crack in a rectangular strip loaded by axial load P and applied moment M

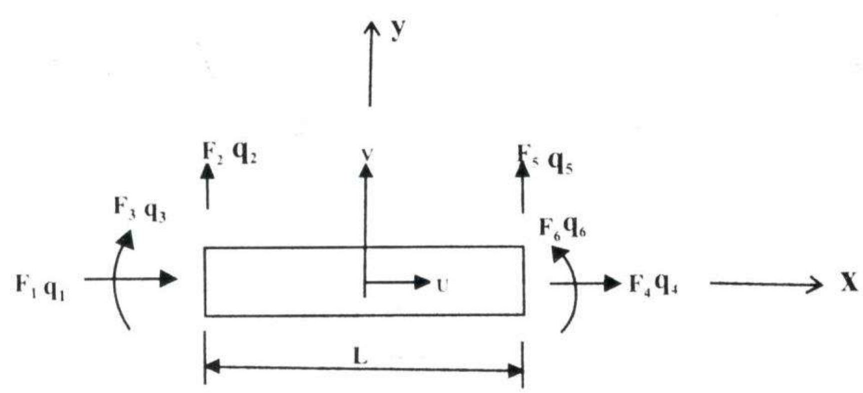


Figure 2. Forces and the displacements bean column element

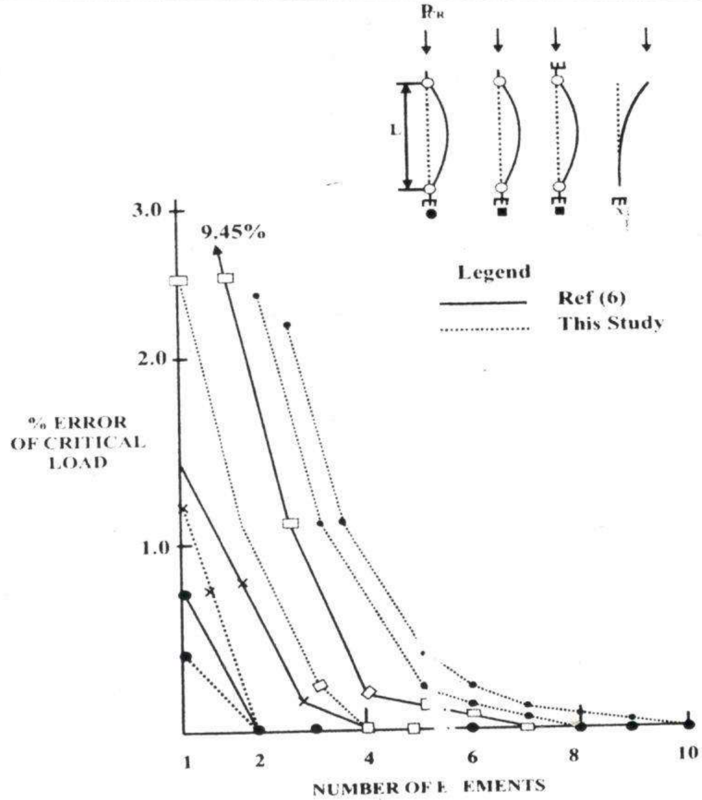
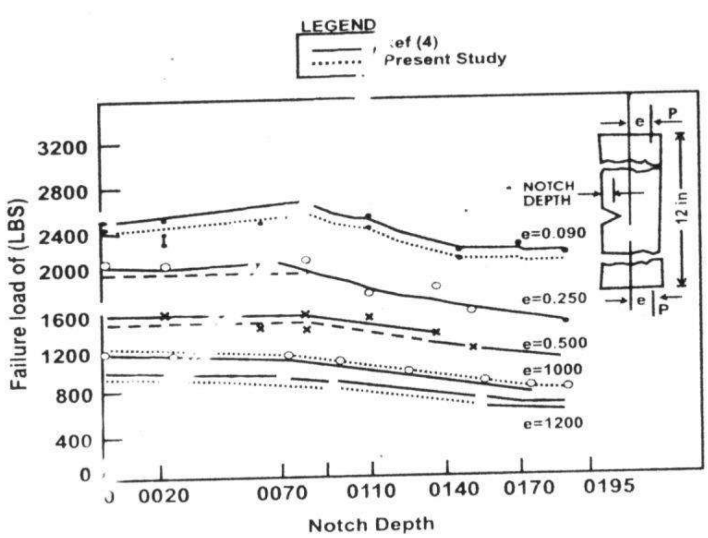
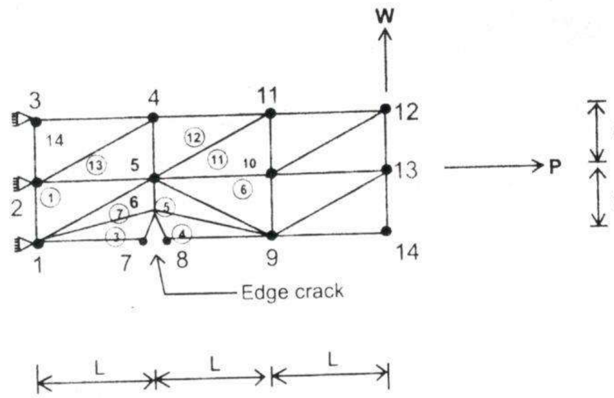


Figure. 3 Convergence of Finite Elemen/ Section for Fletural buckling.



Figu e 4: Effect of depth of notch with a fatigue crack on failure load



b: Cracked Cantilever Model

Figure. 5 **Finite element model for crack sensitivity analysis**

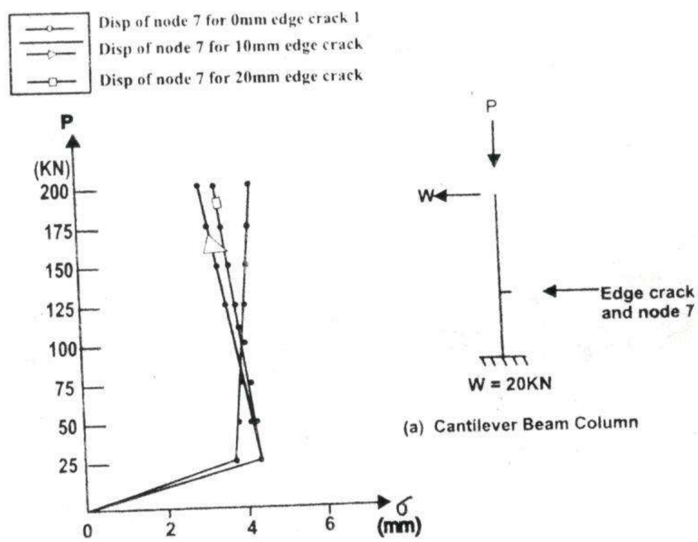


Figure 6 **(b) Sensitivity of cracked cantilever beam column with tip load W. Displacement sensibility of node 7 to crack for cantilever beam column**