NSE TECHINCAL TRANSACTION VOL. 44, N0.4, OCT. - DEC., 2009
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COMPARATIVE ANALYSIS OF TWO-WAY SLABS BY THE FINITE ELEMENT METHOD AND BS 8110

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ABSTRACT

Two basic methods are presently used to analyse slabs spanning in two directions. These are the so-called 'exact' theory of plates, which is based on elastic analysis under service loads, and yield-line theory, in which the behaviour of the slab as collapse approaches is considered. The ultimate bending moment coefficients given in the Codes of Practice CP 110 and BS 8110 are derived from yield-line theory which has been shown to provide only upper-bound solutions. This paper proposes the finite element method as a more versatile tool for the analysis of two-way spanning slabs. The results show that the method is more economical and gives sufficiently accurate solutions compared to the yield-line solutions of the Codes of Practice.

NOTATION

{B} - Matrix relating element strains to element nodal displacements

[D] - Elasticity matrix

D_x , D_y - Bending rigidities of slab

D_{xy}, D_{yx}- Torsional rigidities of slab

D_1, D_2 - Coupling rigidities in the x and y-directions respectively

$[K^e]$ - Element stiffness matrix

$\{\Delta^e\}$ - Vector of nodal displacements

$\varepsilon(x,y)$ - State of strain at a point

$\sigma(x,y)$ - State of stress at a point

θ_x, θ_y - Rotations about x and y-axis respectively

w - Transverse displacement

INTRODUCTION

In practice, the analysis of slabs is frequently complicated by the support conditions. When a slab is supported other than on two opposite sides only, the precise amount and distribution of the load taken by each support, and consequently the magnitude of the bending moments on the slab, are not easily calculated if assumptions resembling practical conditions are made.

Two basic methods are commonly used to analyse slabs spanning in two directions.

These are the theory of plates, which is based on an elastic analysis under service loads, and yield-line theory, in which the behaviour of the slab as collapse approaches is considered. The theory of elastic plates considers the equilibrium of a thin uniform plate under the action of various loads and reactions that are normal to the plane of the undeformed plate. It formulates the continuum problem in terms of fourth-order partial differential equations, the solution of which would provide exact solutions of the stress analysis problem, [1]. However, exact solutions are seldom available for slabs having arbitrary shapes and non-standard support conditions. Numerical methods are therefore conveniently used to analyse such problems.

Johansen's yield-line method requires the designer to postulate first an appropriate collapse mechanism for the slab being considered. Then, if the slab is deformed the energy absorbed in inducing ultimate moments along the yield lines is equal to the work done on the slab by the applied load in producing this deformation. Thus the load determined is the maximum that the slab will support before failure occurs. However, since the method does not investigate the conditions between the postulated yield lines to ensure that the moments in these areas do not exceed the ultimate resistance of the slab, there is no guarantee that the minimum load which may cause collapse has been found, [2]. This is one shortcoming of upper-bound solutions such as those given by Johansen's yield-line theory. The ultimate bending moment coefficients given in CP 110 and BS 8110 [3, 4]are derived from a yield-line analysis in which the coefficients have been adjusted to allow for the non-uniformity of the reinforcement spacing resulting from the division of the slab into middle strips and edge strips, [5].

This paper proposes the use of the finite element method as a more versatile alternative for the analysis of two-way spanning slabs.

FINITE ELEMENT THEORY

The matrix displacement method of analysis is used. The slab is divided into a number of sub-regions, called finite elements, which are assumed to be interconnected at the nodal points only. A suitable displacement function is then chosen for the finite elements in order to approximate the behaviour of the actual structure. Based on the chosen displacement function, equations of equilibrium are developed in terms of the unknowns at the nodal points. The principle of virtual work is applied to derive the individual element stiffness and load matrices. Using superposition, these individual element stiffness and load matrices are assembled to form the global matrices. The resulting algebraic equations are solved for the unknown nodal displacements. Once the displacements are known, the strains may then be evaluated from the strain-displacement relations, and finally the stresses are determined from the stress-strain relations.

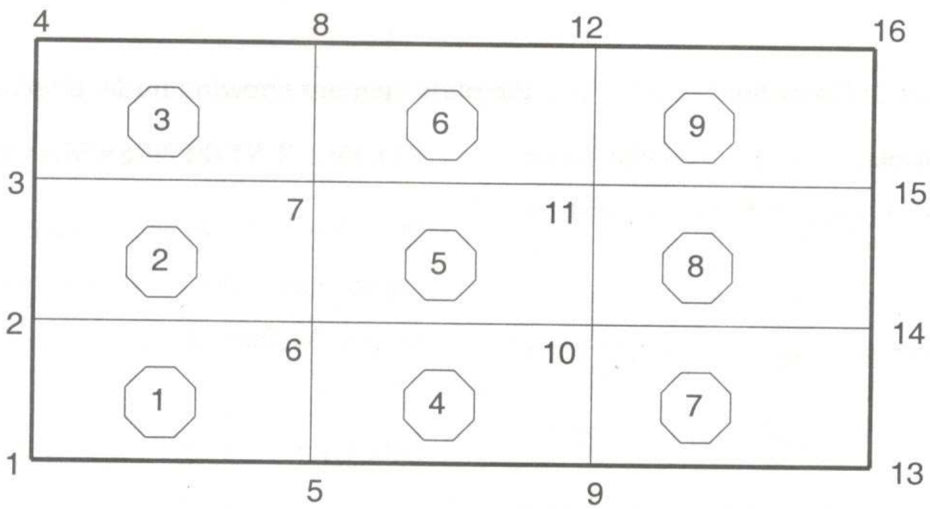


Fig. 1: Slab discretised into rectangular finite elements

The slab is represented by non-conforming but complete rectangular plate bending elements with three degrees of freedom per node (w, θ_x, θ_y) and a cubic displacement model, (Fig. 2). Studies by Gallagher [**6**] showed that this element is efficient and yields solutions of acceptable accuracy.

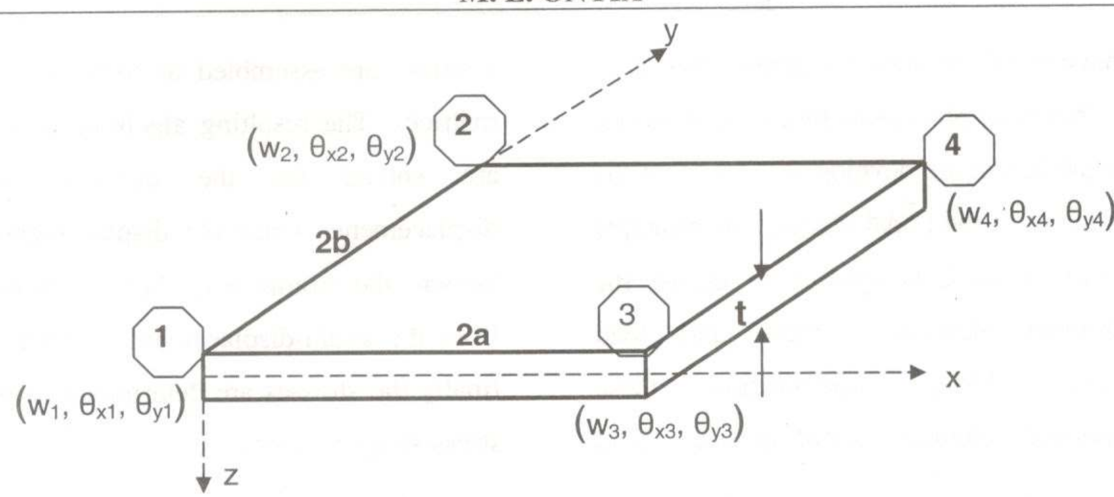


Fig. 2: Co-ordinate system for the plate element showing nodal displacements

A suitable displacement function chosen for the rectangular plate bending element is: [7]

$$W = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3 \quad (1)$$

Note that since the element has twelve degrees of freedom, twelve undetermined constants (α_1 to α_{12}) have been employed in the polynomial expression chosen to represent w.

Although the chosen displacement function involves a discontinuity of cross slope at inter-element boundaries, and is said to be non-conforming, it is nevertheless used for the rectangular plate bending element since it has been found to exhibit good convergence, [7].

ELEMENT STIFFNESS MATRIX

The above displacement function defines displacement $\Delta(x,y)$ at any point in the rectangular finite element:

$$\{\Delta(x,y)\} = \begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} \quad (2)$$

The nodal displacements $\{\Delta^e\}$ can be obtained by substituting the coordinates of the element in $\{\Delta(x,y)\}$.

$$\{\Delta^e\} = \begin{Bmatrix} \Delta(x_1,y_1) \\ \Delta(x_2,y_2) \\ \Delta(x_3,y_3) \\ \Delta(x_4,y_4) \end{Bmatrix} \quad (3)$$

The state of strain at any point $\varepsilon(x,y)$ is expressed by the curvatures and twist:

$$\{\varepsilon(x,y)\} = \begin{Bmatrix} -\partial^2 w/\partial x^2 \\ -\partial^2 w/\partial y^2 \\ 2\partial^2 w/(\partial x\partial y) \end{Bmatrix} \quad (4)$$

$\{\varepsilon(x,y)\}$ can be related to the nodal displacements $\{\Delta^e\}$:

$$\{\varepsilon(x,y)\} = [B]\,\{\Delta^e\} \quad (5)$$

where [B] is (3x12) matrix for the rectangular plate element, [**8**].

The internal `stresses' are the bending and twisting moments. Thus, the state of stress can be represented by:

$$\{\sigma(x,y)\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \quad (6)$$

From the constitutive relationship:

$$\{\sigma(x,y)\} = [D]\,\{\varepsilon(x,y)\} \quad (7)$$

where [D] is the elasticity matrix.

$$[D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \quad (8)$$

The element stiffness matrix $[K^e]$ can be obtained by replacing the internal stresses $\{\sigma(x,y)\}$ with statically equivalent nodal forces $[F^e]$.

By applying the principle of virtual work, we derive the element stiffness matrix to be:

$$[K^e] = \int_V [B]^T[D]\,[B]\,dvol \quad (9)$$

or, for the plate bending finite element, we have that:

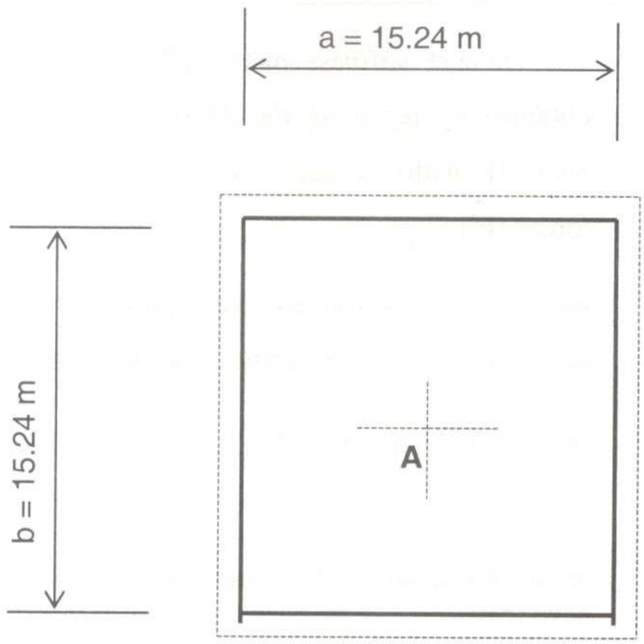
$$[K^e] = \int_A [B]^T[D]\,[B]\,dA \quad (10)$$

The nodal forces are obtained using the same principle of virtual work.

Explicit expressions for the element stiffness matrix for an orthotropic material and the element load vector for the plate bending finite element have been evaluated by Zienkiewicz [**7**].

NUMERICAL EXAMPLES

A square slab of uniform thickness is supported along all edges as shown in Fig. 3. Consider a uniformly distributed load in the analysis.



Modulus of elasticity, $E = 26 \times 10^6$ kN/m^2

Poisson ratio, ν = 0.3

Thickness of plate, t = 0.20 m

Fig.3: All-sides supported square slab under load (A – center of slab)

From the Codes of Practice BS 8110 and CP 110, the bending moments, M_x and M_y, at the centre A under a uniformly distributed load, q, are given by [**3, 4, 9**]:

$M_x = M_y = \alpha q a^2$

The deflection, W, at the centre A under a uniformly distributed load, q, is given by [**3, 4, 9, 10, 11, 12**]:

$W = K (qa^4)/EI$, where $K=5/384$

The results for simple and fixed supports are presented in Tables 1 & 2, and plots of the convergence curves for central moments and deflections for both the simply-supported and fixed cases are presented in Figs 4a and 4d. :

Table1: COMPUTED DEFLECTIONS AND MOMENTS OF A SIMPLY SUPPORTED SQUARE SLAB UNDER UNIFORM LOAD, q

Mesh	Transverse deflection W at A ($\times qa^4/D$)	Bending Moments $M_x = M_y$ at A ($\times qa^2$)
2 x 2	0.00506	0.0660
4 x 4	0.00433	0.0522
6 x 6	0.00418	0.0497
8 x 8	0.00413	0.0489
12 x 12	0.00409	0.0484
16 x 16	0.00408	0.0481
Exact [1]	**0.00406**	**0.0479**
% Difference (b/w Exact and 16x16 mesh)	0.49	0.42
BS 8110 & CP 110	**0.00568**	**0.062**

Table 2: COMPUTED DEFLECTIONS AND MOMENTS OF A FIXED-SUPPORTED SQUARE SLAB UNDER UNIFORM LOAD, q

Mesh	Transverse deflection W at A ($\times qa^4/D$)	Bending Moments $M_x = M_y$ at A ($\times qa^2$)
2 x 2	0.00148	0.04616
4 x 4	0.00140	0.02778
6 x 6	0.00133	0.02496
8 x 8	0.00130	0.02405
12 x 12	0.00128	0.02341
16 x 16	0.00128	0.02319
Exact [1]	**0.00126**	**0.02310**
% Difference (b/w Exact and 16x16 mesh)	1.56	0.39
BS 8110 & CP 110	**0.00149**	**0.0550**

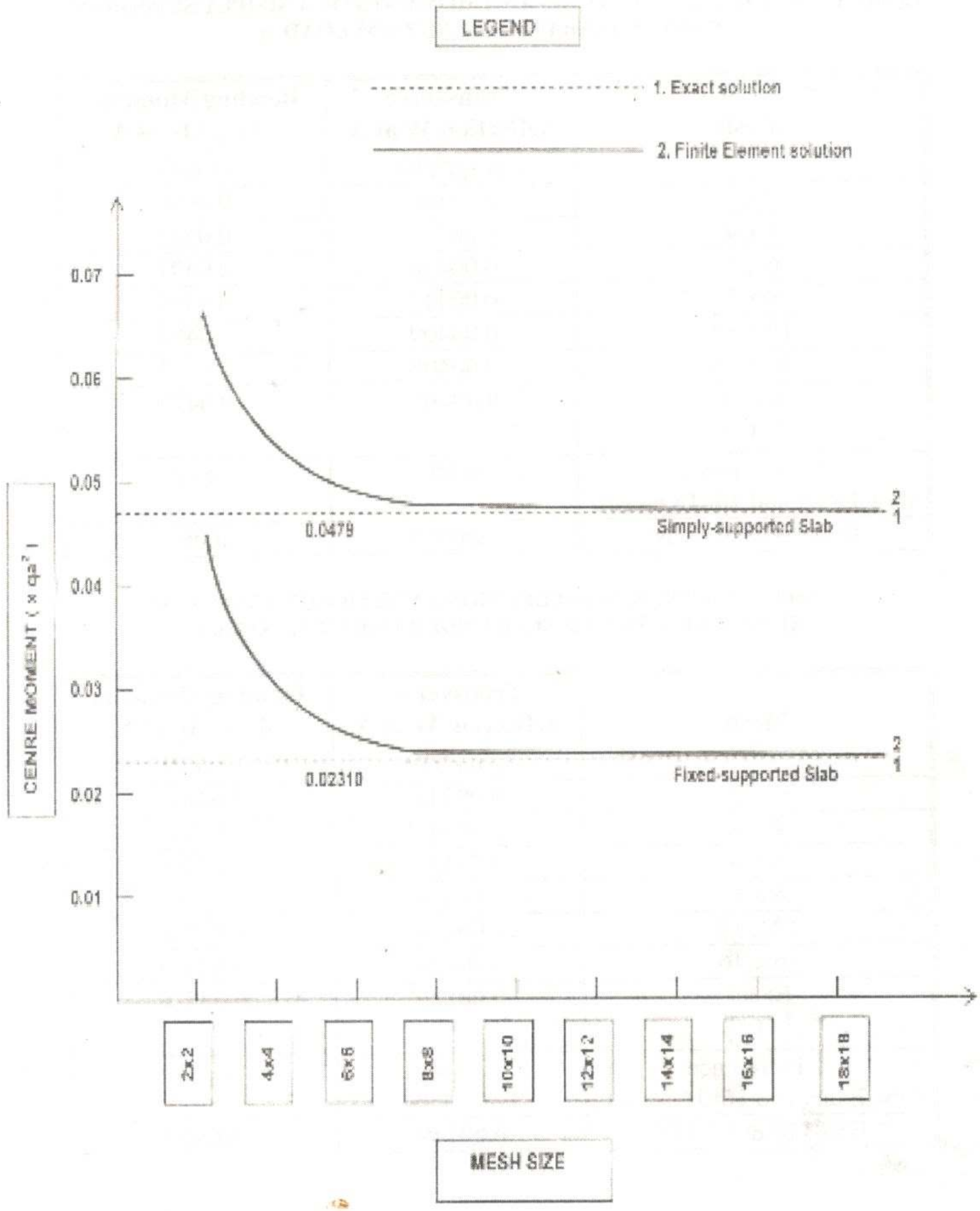


Fig. 4a: Convergence Curves for the Centre Moments of a Square Slab Under Uniform Load

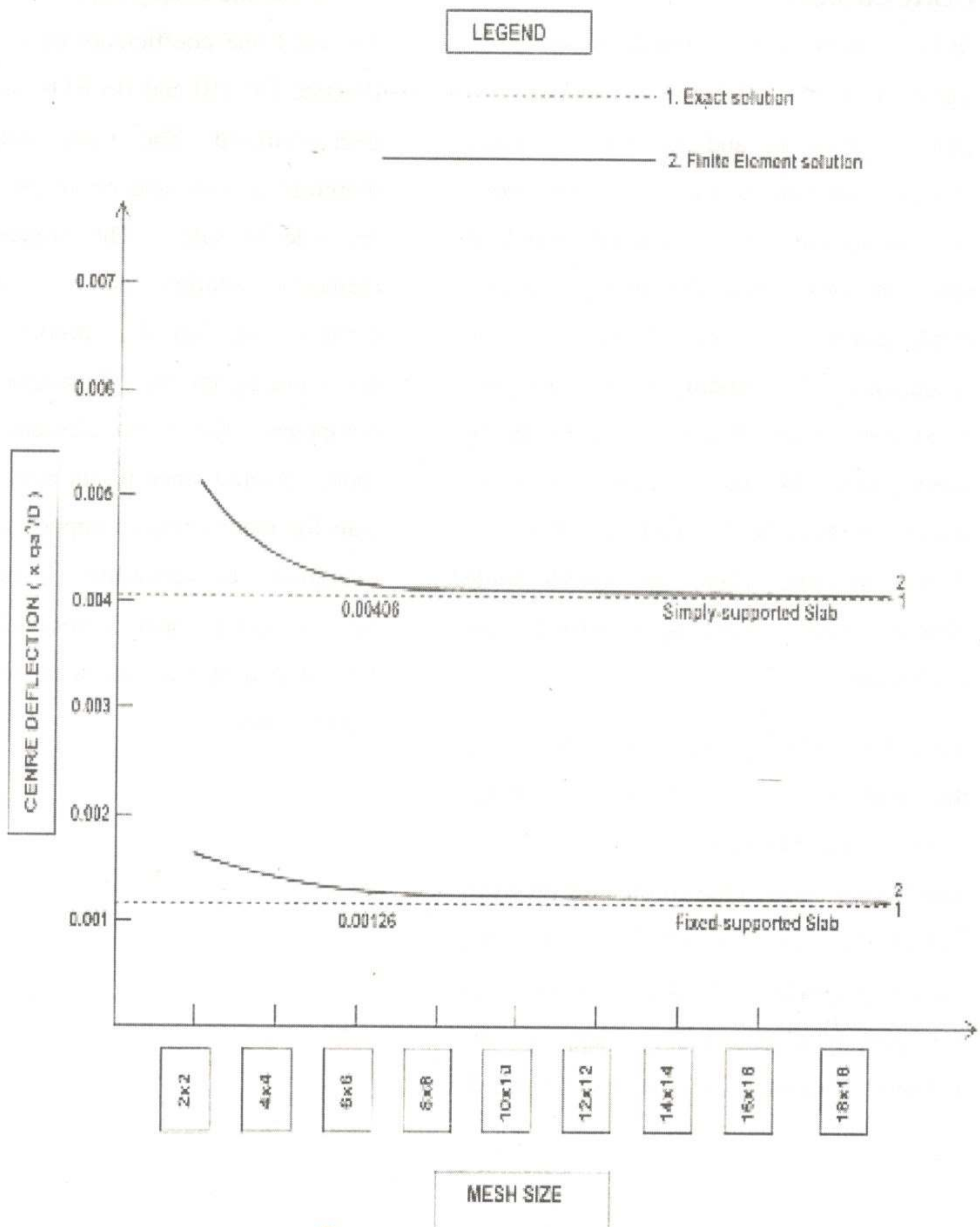


Fig. 4b: Convergence Curves for the Centre Transverse Deflections of a Square Slab Under Uniform Load

DISCUSSION OF RESULTS AND CONCLUSION

It is apparent from the results presented in Tables 1 and 2 and from the convergence plots in Figs 4a and 4b that the finite element solution converges to the exact solution as the mesh is refined, which is sufficient proof that the non-conforming displacement function chosen for the rectangular plate bending element exhibits good convergence. For instance, the 16x16 mesh gives $0.0481qa^2$ as against the exact solution of $0.0479qa^2$ (which is $0.048qa^2$ to 3 dec. places). Thus, the 16x16 finite element mesh yields approximately the exact solution.

Table 1 (simply-supported case) shows that the bending moments M_x and M_y, at the centre A is $0.0660qa^2$ when the mesh is 4x4. This is very close to the solution of BS 8110 and CP 110 which give $0.0620qa^2$. Thus, the solution given by the Codes is equivalent to the finite element solution offered by the 4x4 mesh. Comparing the exact solution and that of the Codes, we observe a difference of 29.4% for the simply-supported slab. Thus, the yield-line coefficients of the Codes of Practice CP 110 and BS 8110 are generally over-estimated. The Code solutions are therefore grossly uneconomical, though on the side of safety. The numerical (finite element) solution is no doubt more cumbersome, but this problem is easily overcome by the use of modern high-speed computers. The finite element method is more versatile since it can also analyse the slab for more complex support and loading conditions. Its versatility can be improved by including shear deformation in the formulation in order to cater for very thick slab situations.

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